Annals of R.S.C.B., ISSN:1583-6258, Vol. 25, Issue 4, 2021, Pages.17627 - 17630
Received 29 January 2021; Accepted 04March 2021.

On Upper and Lower Faintly gζ*-Continuous Multifunctions

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Abstract
The aim of this paper is to introduce and study upper and lower faintly $g\zeta^*$-continuous multifunction in topological spaces.
Keywords: gζ*-open set,gζ*-closed set, faintly gζ*-continuous multifunction.

1. Introduction

It is well known that various type of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared and a good number of them have been extended to the sitting of multifunctions. This implies that both functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. V.Kokilavani et al., introduced the concept of $g\zeta^*$-closed sets in topological spaces[1]. In this paper, we introduce and study upper and lower faintly $g\zeta^*$-continuous multifunction in topological spaces and obtain some characterizations and basic properties of such multifunctions.

2. Preliminaries

Throughout this paper, $(X,\tau)$and (Y,σ)always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let Abe a subset of a space X. For a subset A of $(X,\tau), Cl(A)$ and $Int(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset A is said to be regular closed [2] if $A = Cl(Int))$. A subset N of a topological space (X,τ) is said to be $g\zeta^*$-neighbourhood of a point $x \in X$, if there exists an $g\zeta^*$-open set V such that$x \in V \subset N$. A point $x \in X$ is called a θ-cluster point of A[3] if $Cl(V) \cap A \neq \emptyset$ for every open set V of X containing x.The set of all θ-cluster points of A is called the θ-closure of A and is denoted by $Cl_\theta(A)$.If $A = Cl_\theta(A)$, then A is said to be θ-closed[3]. The complement of a θ-closed set is said to be a θ-open set[3].By a multifunction $F:(X,\tau) \to (Y,\sigma)$, we mean a point-to-set correspondence from X into Y, also we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F:(X,\tau) \to (Y,\sigma)$, the upper and lower inverse of any subset A of Y are denoted by $F^+(A)$ and $F^-(A)$, respectively, that $F^+(A) = \{x \in X: F(x) \subseteq A\}$ and $F^-(A) = \{x \in X: F(x) \cap A \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X: y \in F(x)\}$ for each point $y \in Y$. A multifunction $F:(X,\tau) \to (Y,\sigma)$ is said to be surjective if $F(X) = Y$. A multifunction $F:(X,\tau) \to (Y,\sigma)$ is said to be lower $g\zeta^*$-continuous (resp. upper $g\zeta^*$-continuous) multifunction if $F^-(V) \in g\zeta^*O(X)$ (resp. $F^+(V) \in g\zeta^*O(X)$)for every $V \in \sigma$. A multifunction F is said to be upper(lower) faintly continuous [4] if for each $x \in X$ and for each open set V of Y such that $x \in F^+(V)(x \in F^-(V))$, there exists an open set U of X containing x such that $U \subset F^+\left(Int\left(Cl(V)\right)\right)\left(U \subset F^-\left(Int\left(Cl(V)\right)\right)\right)$. A multifunction$F$ is faintly continuous [4] if it is both upper faintly continuous and lower faintly continuous.

3. Faintly $g\zeta^*$-Continuous Multifunctions:

Definitions 3.1.

A multifunction $F:(X,\tau) \to (Y,\sigma)$ is said to be

(i) Upper faintly $g\zeta^*$-continuous at $x \in X$ if for each θ-open set V of Y containing $F(x)$, there exists $U \in g\zeta^*O(X)$ containing x such that $F(U) \subset V$.

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(ii) Lower faintly $g\zeta^*$-continuous at $x \in X$ if for each θ-open set V of Y such that $F(x) \cap V \neq \emptyset$,there exists $U \in g\zeta^*O(X)$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.

(iii) upper(lower) faintly $g\zeta^*$-continuous if it has this propertyat each point of X.

Remark 3.2. It is clear that every upper $g\zeta^*$-continuous multifunction is upper faintly $g\zeta^*$-continuous.But the converse if not true in general, as the following example shows.

Example 3.3. Let $X = \{a, b, c\}, \tau = \{X, \emptyset, \{b\}\}, \sigma = \{X, \emptyset, \{a\}\}$.Then the multifunction $F: (X, \tau) \to (X, \sigma)$ defined by $F(x) = \{x\}$ is upper faintly $g\zeta^*$-continuous but it is not upper $g\zeta^*$-continuous.

Theorem 3.4.

For a multifunction $F: (X, \tau) \to (Y, \sigma)$, the following are equivalent:

(i) F is upper faintly $g\zeta^*$-continuous.

(ii) For each $x \in X$ and for each θ-open set V such that $x \in F^+(V)$,there exists a $g\zeta^*$-open set U containing x such that $U \subset F^+(V)$.

(iii) For each $x \in X$ and for each θ-closed set V such that $x \in F^+(Y - V)$,there exists a $g\zeta^*$-closed set G such that $x \in X - G$ and $F^-(V) \subset G$.

(iv) $F^+(V)$ is $g\zeta^*$-open for any θ-open set V of Y.

(v) $F^-(V)$ is $g\zeta^*$-closed for any θ-closed set V of Y.

(vi) $F^-(Y - V)$ is $g\zeta^*$-closed for any θ-open set V of Y.

(vii) $F^+(Y - V)$ is $g\zeta^*$-open for any θ-closed set V of Y.

Proof:

(i)⇔(ii)

Let F is upper faintly $g\zeta^*$-continuous, let $x \in X$ and V be aθ-open set of Y such that $F(x) \subseteq V$ then $x \in F^+(V)$ there exists an $g\zeta^*$-open set U containing x such that $F(U) \subset V$ which implies $U \subset F^+(V)$.

(ii)⇔(iii)

Let $x \in X$ and V be a θ -closed set of Y such that $x \in F^+(Y - V)$. By (ii), there exists a $g\zeta^*$-open set U containing x such that $U \subset F^+(Y - V)$. Thus $F^-(V) \subset X - U$.Take $G = X - U$. Then $x \in X - G$ and G is $g\zeta^*$-closed. The converse is similar.

(iv)⇔(v)

Let V be any θ -closed set of Y. Then $Y \setminus V$ is an θ-open set. By (iv),$F^+(Y \setminus V) = X \setminus g\zeta^*(F^-(V))$. Thus $g\zeta^*(F^-(V)) \subset F^-(V)$.Therefore $F^-(V)$is $g\zeta^*$-closed for any θ-closed set V of Y.

(i)⇔(iv)

Let $x \in F^+(V)$ and V be a θ-open set of Y. By (i), there exists a $g\zeta^*$-open set U_x containing x such that $U_x \subset F^+(V)$. Thus, $F^+(V) = \bigcup_{x \in F^+(V)} U_x$. Since any union of $g\zeta^*$-open sets if $g\zeta^*$-open, $F^+(V)$ is $g\zeta^*$-open. The converse is clear.

Theorem 3.5.

For a multifunction $F: (X, \tau) \to (Y, \sigma)$, the following are equivalent:

(i) F is lower faintly $g\zeta^*$-continuous.

(ii) For each $x \in X$ and for each θ-open set V such that $x \in F^-(V)$,there exists a $g\zeta^*$-open set U containing x such that $U \subset F^-(V)$.

(iii) For each $x \in X$ and for each θ-closed set V such that $x \in F^-(Y - V)$,there exists a $g\zeta^*$-closed set G such that $x \in X - G$ and $F^+(V) \subset G$.

(iv) $F^-(V)$ is $g\zeta^*$-open for any θ-open set V of Y.

(v) $F^+(V)$ is $g\zeta^*$-closed for any θ-closed set V of Y.

(vi) $F^+(Y - V)$ is $g\zeta^*$-closed for any θ-open set V of Y.

(vii) $F^-(Y - V)$ is $g\zeta^*$-open for any θ-closed set V of Y.

Proof:

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(i)⇔(ii)

Let $x \in X$ and V be a θ-open set of Y such that $F(x) \cap V \neq \emptyset$ then there exists an $g\zeta^*$-open set U containing x such that $F(u) \cap V \neq \emptyset$ which implies $U \subset F^-(V)$.

(ii)⇔(iii)

Let $x \in X$ and V be a θ -closed set of Y such that $x \in F^-(Y - V)$. By (ii), there exists a $g\zeta^*$-open set U containing x such that $U \subset F^-(Y - V)$. Thus $F^+(V) \subset X - U$.Take $G = X - U$. Then $x \in X - G$ and G is $g\zeta^*$-closed. The converse is similar.

(iv)⇔(v)

Let V be any θ -closed set of Y. Then $Y \setminus V$ is an θ-open set. By (iv),$F^-(Y \setminus V) = X \setminus g\zeta^*\left(F^+(V)\right)$. Thus $g\zeta^*\left(F^+(V)\right) \subset F^+(V)$. Therefore $F^+(V)$is $g\zeta^*$-closed for any θ-closed set V of Y.

(i)⇔(iv)

Let $x \in F^-(V)$ and V be a θ-open set of Y. By (i), there exists a $g\zeta^*$-open set U_x containing x such that $U_x \subset F^-(V)$. Thus, $F^-(V) = \bigcup_{x \in F^-(V)} U_x$. Since any union of $g\zeta^*$-open sets if $g\zeta^*$-open, $F^-(V)$ is $g\zeta^*$-open. The converse is clear.

Definition 3.6.

For a multifunction $F: (X, \tau) \to (Y, \sigma)$, the graph multifunction $G_F: X \to X \times Y$ is defined as, $G_F(x) = \{x\} \times \{F(x)\}$ for every $x \in X$ and the subset of $\{\{x\} \times F(x): x \in X\} \subset X \times Y$ is called the graph multifunction of F is denoted by $G(x)$.

Lemma 3.7.

For a multifunction $F: (X, \tau) \to (Y, \sigma)$, the following holds:

(i) ${G_F}^+(A \times B) = A \cap F^+(B)$ and

(ii) ${G_F}^-(A \times B) = A \cap F^-(B)$

For each subsets $A \subset X$ and $B \subset Y$.

Theorem 3.8.

Let $F: (X, \tau) \to (Y, \sigma)$ be a multifunction. If the graph multifunction of F is an upper faintly $g\zeta^*$-continuous, then F is upper faintly $g\zeta^*$-continuous.

Proof:

Let $x \in X$ and V be any θ -open subset of Y such that $x \in F^+(V)$.We obtain that $x \in {G_F}^+(X \times V)$ and that $X \times V$ is a θ–open set. Since the graph multifunction G_F is upper faintly $g\zeta^*$-continuous, it follows that there exists an $g\zeta^*$-open set U of X containing x such that $U \subset {G_F}^+(X \times V)$. Since $U \subset {G_F}^+(X \times V) = X \cap F^+(V) = F^+(V)$. We obtain that $U \subset F^+(V)$. Thus F is upper faintly $g\zeta^*$-continuous.

Theorem 3.9.

A multifunction $F: (X, \tau) \to (Y, \sigma)$ is lower faintly $g\zeta^*$-continuous if $G_F: X \to X \times Y$ is lower faintly $g\zeta^*$-continuous.

Proof:

Suppose that G_F is lower faintly $g\zeta^*$-continuous. Let $x \in X$ and V be any θ-open set of Y such that $x \in F^-(V)$. Then $X \times V$ is a θ–open in $X \times Y$ and $G_F(x) \cap (X \times V) = \left(\{x\} \times F(x)\right) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is lower faintly $g\zeta^*$-continuous, there exists an $g\zeta^*$-open U containing x such that $U \subset {G_F}^-(X \times V)$. Hence $U \subset F^-(V)$.This shows that F is lower faintly $g\zeta^*$-continuous.

Theorem 3.10.

Suppose that (X, τ) and (X_α, τ_α) are topological spaces where $\alpha \in J$. Let $F: X \to \prod_{\alpha \in J} X_\alpha$ be a multifunction from X to the product space $\prod_{\alpha \in J} X_\alpha$ and Let $P_\alpha: \prod_{\alpha \in J} X_\alpha \to X_\alpha$ be the projection multifunction for each $\alpha \in J$ which is defined by $P_\alpha(x_\alpha) = \{x_\alpha\}$. If F is an upper(lower) faintly $g\zeta^*$-continuous multifunction, then $P_\alpha \circ F$ is an upper(lower) faintly $g\zeta^*$-continuous multifunction for each $\alpha \in J$.

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Proof:

Take any $\alpha_0 \in J$. Let V_{α_0} be a θ-open set in $(X_{\alpha_0}, \tau_{\alpha_0})$. Then $(P_{\alpha_0} \circ F)^+(V_{\alpha_0}) = F^+\left({P_{\alpha_0}}^+(V_{\alpha_0})\right) = F^+(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$. Since F is an upper faintly $g\zeta^*$-continuous multifunction and since $V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is a θ-open set, it follows that $F^+(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ is a $g\zeta^*$-open set in (X, τ). This shows that $P_\alpha \circ F$ is an upper faintly $g\zeta^*$-continuous multifunction. Hence, we obtain that $P_{\alpha_0} \circ F$ is an upper faintly $g\zeta^*$-continuous multifunction for each $\alpha \in J$.

Theorem 3.11.

Suppose that for each $\alpha \in J, (X_\alpha, \tau_\alpha), (Y_\alpha, \tau_\alpha)$ are topological spaces. Let $F_\alpha: X_\alpha \to Y_\alpha$ be a multifunction for each $\alpha \in J$ and let $F: \prod_{\alpha \in J} X_\alpha \to \prod_{\alpha \in J} Y_\alpha$ be defined by $F(x_\alpha) = \prod_{\alpha \in J} F_\alpha(x_\alpha)$ from the product space $\prod_{\alpha \in J} X_\alpha$ to the product space $\prod_{\alpha \in J} Y_{\alpha.}$ If F is an upper faintly $g\zeta^*$-continuous multifunction, then each F_α is an upper faintly $g\zeta^*$-continuous multifunction for each $\alpha \in J$.

Proof:

Let V_α be a θ-open set in Y_α.Then$V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta$ is a θ-open set. Since F is an upper faintly $g\zeta^*$-continuous multifunction, it follows that $F^+(V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = {F_\alpha}^+(V_\alpha) \times \prod_{\alpha \neq \beta} X_\beta$is an $g\zeta^*$-open set. Consequently, we obtain that ${F_\alpha}^+(V_\alpha)$ is an $g\zeta^*$-open set. Thus, we show that $F_\alpha$ia an upper faintly $g\zeta^*$-continuous multifunction.

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