

2-Domination Number in Special Classes of Graphs

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ABSTRACT

A set $S \subseteq V$ is a k -dominating set in $G(V, E)$ if every vertex in V not in S has at least k neighbours in S . The k -domination number of G , denoted by $\gamma_k(G)$, is the minimum cardinality of a k -dominating set of G . When $k = 2$, k -dominating set is called 2-dominating set. A dominating set S with the additional property that the subgraph induced by S contains a perfect matching is called a paired-dominating set. The k -domination number and paired domination number of G are the minimum cardinality of a k -dominating set and the minimum cardinality of a paired dominating set respectively of G . In this paper, we obtain 2-dominating number and paired domination number for special classes of graphs such as path necklace, cycle necklace and complete necklace networks. Further, we determine a lower bound for 2-dominating number of special classes of graphs.

KEYWORDS

Dominating set, 2-dominating Set, P-necklace, C-necklace, K-necklace, Star Necklace Graphs.

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Introduction

The domination in graphs is one of the concepts in graph theory which has attracted many researchers to work on it because it has the potential to solve many real life problems involving design and analysis of communication network as well as defense surveillance. Many variants of domination models are available in the existing literature such as independent domination, total domination, connected domination, edge domination and so on. In this paper, we obtain 2-dominating number and paired domination number for special classes of graphs.

For $v \in V(G)$, the open neighbourhood of v , denoted as $N_G(v)$, is the set of vertices adjacent with v ; and the closed neighbourhood of v , denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighbourhood of S is defined as $N_G(S) = \bigcup_{v \in S} N_G(v)$ and the closed neighbourhood of S is defined as $N_G[S] = N_G(S) \cup S$. For brevity, we denote $N_G(S)$ by $N(S)$ and $N_G[S]$ by $N[S]$. For a graph $G(V, E)$, $S \subseteq V$ is a dominating set of G if every vertex in $V \setminus S$ has at least one neighbour in S . The domination number of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G . A set $S \subseteq V$ is a k -dominating set in $G(V, E)$ if every vertex in V not in S has at least k neighbours in S . The k -domination number of G , denoted by $\gamma_k(G)$, is the minimum cardinality of a k -dominating set of G . When $k = 2$, k -dominating set is called 2-dominating set. The 2-dominating number of G is the minimum cardinality of a 2-dominating set of G .

If no two edges in it are adjacent to G , that is, an independent edge set is a set of edges without common vertices, the set of edges in the graph G is independent. A matching G in a graph is a group of independent G edges. The matching graph number of G denoted by $\alpha'(G)$, is the maximum matching cardinality of G . A perfect matching of M is a matching of any vertex of M . A graph G paired-dominant set is a S dominant set of G , with the additional property that the $G[S]$ subgraph induced by S contains a perfect M match (not necessarily induced). The paired-domination number of G , denoted by $\gamma_{pr}(G)$, is the minimum paired-dominant cardinality set in G .

In 1985 Fink and Jacobson [2] introduced k -dominating set and also shown that $\gamma_k(G) \geq \gamma(G) + k - 2$. Haynes, Hedetniemi, and Slater [5], as well as to the excellent survey on k -domination in graphs by Chellali, Favaron, Hansberg, and Volkmann [3] and so on.

Main Results

In this section, we prove that the 2-domination number and pairedomination number are equal for some special classes of graphs like P-necklace, C-necklace and K-necklace and star necklace graphs and we obtain a lower bound for 2-domination number with respect to cut-vertices.

In 2019 Anitha et.al.[7] proved that a lower bound for the domination number of a graph G and is quoted below. With this connection we have given a lower bound for 2-domination number of a graph G .

Lemma 2.1:[7] Let G be a graph on n -vertices with cut vertices $v_1, v_2, \dots, v_m, m \geq 2$. Suppose for every cut vertex v_i of G , there exists components $H_{i1}, H_{i2}, \dots, H_{ik}, K_i \geq 1$, such that $G[(V(H_{ij}) \cup v_i)]$ is a complete graph on r_{ij} vertices, $r_{ij} \geq 3, 1 \leq j \leq k_i, 1 \leq i \leq m$. Then $\gamma(G) \geq m$.

A critical subgraph of H of G is defined by the following lemma in the sense that H contains at least two vertices of any 2-dominating set.

Lemma 2.2: Let G be a graph and H as shown in Figure 1(a) be a subgraph G . Then H is a critical subgraph of G .

Proof. Let S be a 2-dominating set of G . We claim that $|S|=2$. Suppose not, let us assume that $|S|=1$. Let $V(H) = \{K_{t_i}\}$ as shown in Figure 1(a).

We note that H is connected to the rest of the graph only through the cut vertices. say $v_i, 1 \leq i \leq m$. Now we have the following two cases:

1. $S \cap V(H) \neq \emptyset$.
2. $S \cap V(H) = \emptyset$.

In both these cases, the $V(H)$ are not 2-dominated, a contradiction.

Lemma 2.3: Let G be a graph on n -vertices with cut vertices $v_1, v_2, \dots, v_m, m \geq 2$. Suppose for every cut vertex v_i of G , there exists components $H_{i1}, H_{i2}, \dots, H_{ik}, C_i \geq 3$, such that $G[(V(H_{ij}) \cup v_i)]$ is a cycle on r_{ij} vertices, $r_{ij} \geq 3, 1 \leq j \leq n, 1 \leq i \leq m$. Then $\gamma_2(G) \geq m \frac{n}{2}$.

Proof. We claim that $|S| = m \frac{n}{2}$. Let $H_i, 1 \leq i \leq m$ be the components of $G \setminus v_i$ such that $G[H_i \cup v_i] \cong C_{t_i}$. There are m vertex disjoint copies of C_{t_i} in G . Let S be the 2-dominating set which contains v_i s of every C_{t_i} . Let H is isomorphic to cycle C_j . Then the vertices of H is connected to through only $v_i, i = 1, 2, \dots, m$. Let us assume the contrary that there exist a H such that $V(H) \cap S = \frac{n}{2} - 1$. Then at least one vertex in H does not 2-dominate any member of S , a contradiction. This implies that, $\gamma_2(G) \geq m \frac{n}{2}$.

P-Necklace

Definition 4. [6] Let P_m be a path on m vertices and let K_{t_i} be complete graphs on t_i vertices, $1 \leq i \leq m$. The graph obtained by identifying any one vertex of K_{t_i} with i^{th} vertex of $P_m, 1 \leq i \leq m$, is called a P-necklace and is denoted by $PN(P_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$.

Definition 5. [6] Let C_m be a cycle on m vertices and let K_{t_i} be complete graphs on t_i vertices, $1 \leq i \leq m$. The graph obtained by identifying any one vertex of K_{t_i} with i^{th} vertex of $C_m, 1 \leq i \leq m$, is called a C-necklace and is denoted by $CN(C_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$. See Figure 1(a).

Lemma 2.1.1: Let G be a C-necklace and is denoted by $CN(C_m; K_{t_1}, K_{t_2}, \dots, K_{t_m}), m \geq 4$. Then $\gamma_2(G) \geq 2m$.

Proof. In $CN(C_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$, there are m vertex disjoint copies of H as described in Lemma 2.1. Therefore, $\gamma_2(G) \geq 2m$.

The Algorithm given below computes the 2- domination number in P- Necklace graph $PN(P_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$.

2-Domination Algorithm in P- Necklace

Input: Let G be a P-necklace graph $PN(P_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$.

Algorithm: Label the vertices of $PN(P_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$ as follows:

- (i) Label the vertices of the subgraph P_m of G as v_1, v_2, \dots, v_m from left to right such that each complete graph K_{t_i} shares exactly one vertex of v_i of $P_m, 1 \leq i \leq m$.
- (ii) Let $H_i, 1 \leq i \leq m$ be the components of $G \setminus v_i$ such that $G[H_i \cup v_i] \cong K_{t_i}$ and select an edge $e_i = (u_i, v_i), 1 \leq i \leq m$ from each H_i in S .

Output: $\gamma_2(G) = 2m$.

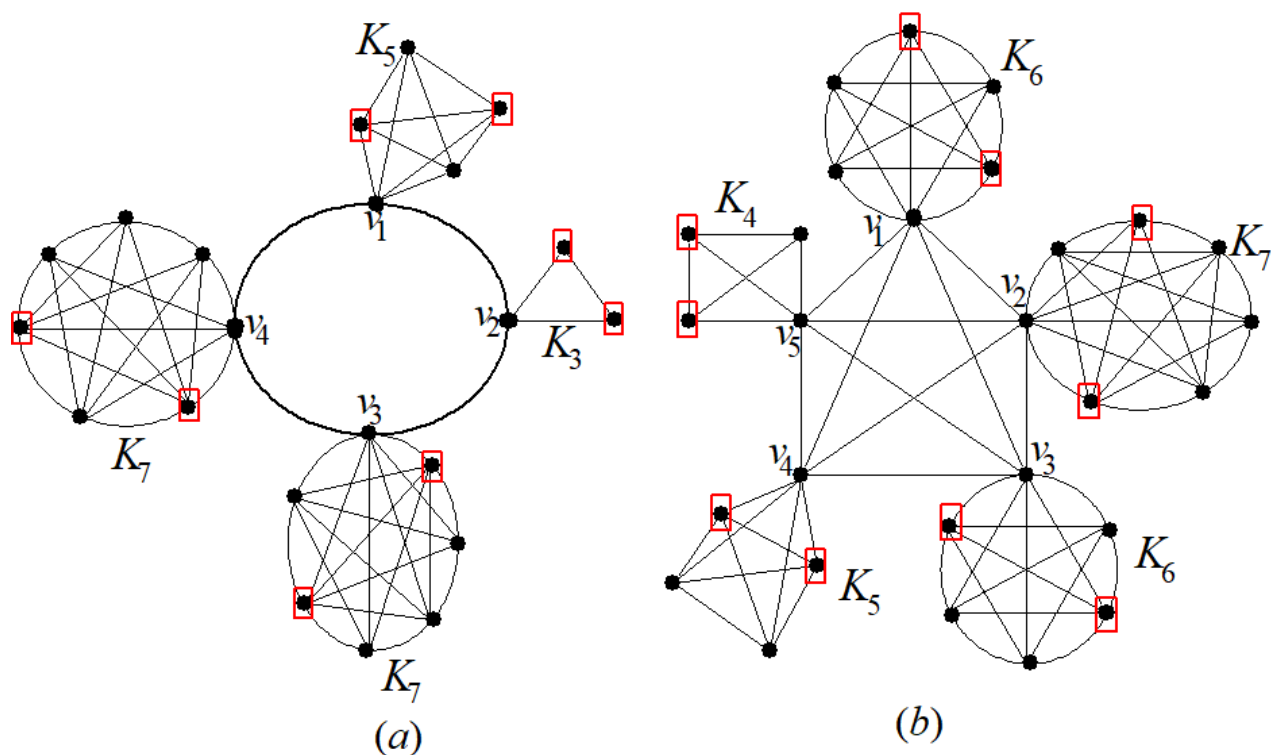


Figure 1. Red colored squared vertices constitutes the 2- dominating set of (a) C-Necklace $N(P_m, K)$ (b) K-Necklace $N(K_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$

Proof of correctness: Let S be the set of $2m$ vertices chosen from each H_i of G . We claim that, $|S| = 2m$. Let v_1, v_2, \dots, v_m be the vertices of the subgraph P_m of G such that each complete graph K_{t_i} shares exactly one vertex of v_i of $P_m, 1 \leq i \leq m$. It is easy to see that v_i is a cut vertex of $G, 1 \leq i \leq m$. Let H_i be the components of $G \setminus v_i$ such that $G[H_i \cup v_i] \cong K_{t_i}$. To prove each vertex in P-necklace is dominated, it is enough to prove that the vertices considered in S dominates the each of $[H_i \cup v_i] \cong K_{t_i}, 1 \leq i \leq m. |S| = 2m$. Moreover, the vertices in S are pairwise adjacent. This implies that $\gamma_2(G) = \gamma_{pr}(G) = 2m$.

Theorem 1. Let G be a P-necklace graph

$PN(P_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$ or a C – necklace graph $CN(C_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$, $n \geq 3$, $t_i \geq 3$, $1 \leq i \leq m$ $onn = \sum_{i=1}^m t_i$ vertices. Then $\gamma_2(G) = 2m$

K- Necklace

Definition 6. [6] Let K_m and K_{t_i} be complete graphs on m and t_i vertices respectively, $1 \leq i \leq m$. The graph obtained by identifying any one vertex of K_{t_i} with the i^{th} vertex of K_m , $1 \leq i \leq m$, is called a K -necklace and is denoted by $KN(K_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$. See Figure 3.

Definition 7. [6] Let $K_{1,m}$ be a star graph on $m + 1$ vertices and let K_{t_i} be a complete graphs on t_i vertices, $1 \leq i \leq m$. The graph obtained by identifying any one vertex of K_{t_i} with i^{th} vertex of $K_{1,m}$, $1 \leq i \leq m$, is called a star necklace and is denoted by $N(K_{1,m}; K_{t_1}, K_{t_2}, \dots, K_{t_m})$ See Figure 1(b).

2- Domination Algorithm in K-Necklace Graph

Input: The K -necklace $KN(K_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$.

Algorithm: Label the vertices of $KN(K_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$ as follows:

- (i) Label the vertices of the subgraph K_m of G as v_1, v_2, \dots, v_m in the clock wise direction such that each complete graph K_{t_i} shares exactly one vertex of v_i of K_m , $1 \leq i \leq m$.
- (ii) Let H_i , $1 \leq i \leq m$ be the components of $G \setminus v_i$ such that $G[H_i \cup v_i] \cong K_{t_i}$, and select 2- vertices from each H_i in S .

Output: $\gamma_2(G) = 2m$

Proof of Correctness: Let S be the set of $2m$ vertices chosen from each H_i of G . We claim that, $|S| = 2m$. Let v_1, v_2, \dots, v_m be the vertices of the subgraph K_m of G such that each complete graph K_{t_i} shares exactly one vertex of v_i of K_m , $1 \leq i \leq m$. It is easy to see that v_i is a cut vertex of G , $1 \leq i \leq m$. Let H_i be the components of $G \setminus v_i$ such that $G[H_i \cup v_i] \cong K_{t_i}$. To prove each vertex in K –necklace is dominated, it is enough to prove that the vertices considered in S dominates the each of $(H_i \cup v_i) \cong K_{t_i}$, $1 \leq i \leq m$. By Lemma 2, domination number of K_{t_i} is 2. There are m distinct components of H in G . Moreover, the vertices in S are pairwise adjacent. Therefore, $|S| = 2m$. This implies that $\gamma_2(G) = \gamma_{pr}(G) = 2m$.

Theorem 2. Let G be the K –necklace $KN(K_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$ or a star necklace $KN(K_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$, $m \geq 4, t_i \geq 4, 1 \leq i \leq m$. Then $\gamma_2(G) = \gamma_{pr}(G) = 2m$.

2-Domination Number of a Graph G, $H_i \cong C_n$

In this section, we solve 2- domination number for special classes of graphs for which each $H_i \cong C_n$

Definition 6. Let P_m be a path on m vertices and let C_{t_i} be cycle on t_i vertices, $1 \leq i \leq n$. The graph obtained by identifying any one vertex of C_{t_i} with i^{th} vertex of P_m , $1 \leq i \leq n$, is called a C -necklace and is denoted by $PN(P_m; C_{t_1}, C_{t_2}, \dots, C_{t_m})$. Let C_m be a cycle on m vertices and let C_{t_i} be complete graphs on t_i vertices, $1 \leq i \leq n$. The graph obtained by identifying any one vertex of C_{t_i} with i^{th} vertex of C_m , $1 \leq i \leq n$, is called a C -necklace and is denoted by $CN(C_m; C_{t_1}, C_{t_2}, \dots, C_{t_m})$. See Figure 2.

Definition 7. [6] Let $K_{1,m}$ be a star graph on $m + 1$ vertices and let C_{t_i} be a cycle on t_i vertices, $1 \leq i \leq n$. The graph obtained by identifying any one vertex of C_{t_i} with i^{th} vertex of $K_{1,m}$, $1 \leq i \leq n$, is called a star necklace and is denoted by $N(K_{1,m}; C_{t_1}, C_{t_2}, \dots, C_{t_m})$. See Figure 3.

2-Domination Algorithm in Star- Necklace

Input: Let G be a P -necklace graph by $N(K_{1,m}; C_{t_1}, C_{t_2}, \dots, C_{t_m})$.

Algorithm: Label the vertices of $PN(K_{1,m}; C_{t_1}, C_{t_2}, \dots, C_{t_m})$ as follows:

- (i) Label the root vertex of $K_{1,m}$ as v_0 .
- (ii) Label the vertices that are adjacent to root vertex v_0 consecutively v_1, v_2, \dots, v_m in the clock wise direction.
- (iii) Let $H_i, 1 \leq i \leq m$ be the components of $G \setminus v_i$ such that that $G[H_i \cup v_i] \cong C_{t_i}$. and select $\frac{n}{2}$ vertices from each H_i in S .

Output: $\gamma_2(G) = m \frac{n}{2}$

Proof of Correctness: Let S be the set of $m \frac{n}{2}$ vertices chosen from each H_i of G . We claim that, $|S| = m \frac{n}{2}$. Let v_1, v_2, \dots, v_m be the vertices of the subgraph $K_{1,m}$ of G such that each cycle C_{t_i} shares exactly one vertex of v_i of $K_{1,m}, 1 \leq i \leq m$. It is easy to see that v_i is a cut vertex of $G, 1 \leq i \leq m$. Let H_i be the components of $G \setminus v_i$ such that $G[H_i \cup v_i] \cong C_{t_i}$. To prove each vertex in star necklace is 2- dominated, it is enough to prove that the vertices considered in S 2-dominates the each of $(H_i \cup v_i) \cong C_{t_i}, 1 \leq i \leq m$. Lemma 3, domination number of C_{t_i} is $m \frac{n}{2}$. Therefore, $|S| = 2m$. This implies that $\gamma_2(G) = m \frac{n}{2}$.

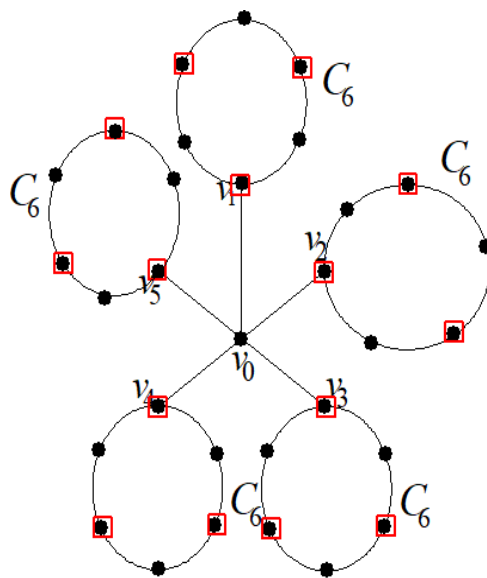


Figure 3. Red colored squared vertices constitutes the 2- dominating set of Star-Necklace $KN(K_{1,5})$

Theorem 3. Let G be the K -necklace $KN(K_m; C_{t_1}, C_{t_2}, \dots, C_{t_m})$ or a star necklace $KN(K_{1,m}; C_{t_1}, C_{t_2}, \dots, C_{t_m})$, $m \geq 4, t_i \geq 4, 1 \leq i \leq m$. Then $\gamma_2(G) = m \frac{n}{2}$.

Theorem 4. Let G be the P -necklace $PN(P_m; C_{t_1}, C_{t_2}, \dots, C_{t_m})$ or a C -necklace $CN(C_n; C_{t_1}, C_{t_2}, \dots, C_{t_m})$, $m \geq 4, t_i \geq 4, 1 \leq i \leq m$. Then $\gamma_2(G) = m \frac{n}{2}$.

Conclusion

In this paper, we 2-domination number for P -Necklace, C -Necklace, K -Necklace and Star Necklace. Further, we obtained a lower bound for 2-domination number with respect to cut vertices. Moreover, we shown that 2-

domination number and paired domination number are equal for P -Necklace, C -Necklace, K -Necklace and Star Necklace, where $H_i \cong K_n$.

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