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Modeling of Recovering of tThe Internal Structure Objects on a Families of Parabolic and Polygonal Lines

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Abstract
We considered a problem of reconstruction function in a strip, if we know the integrals of sought function onthe families of parabolic and polygonal lines with a given weight function of a special kind. We obtained an analytical representation of solution in the class of smooth finite functions, the uniqueness of theorems for solution of the problem is proved. The stability estimate of solution in Sobolev spaces is presented, which implies its weakly ill-posedness. These theoretical results are used in order to recover the internal structure objects from their integral data. Numerical results and their graphical representation are given.
Keywords:Integral geometry problem, Fourier transform, inversion formulae, Tikhonov regularization, Radon transform.

1. INTRODUCTION

There are many applications that deal with recovering the internal structure of objects without harming it. The basic mathematical idea, common to many such reconstruction problems, is based on integral geometry [7]. In the general form, the problem of integral geometry can be formulated as follows. Let $D-$ be a domain of n dimensional space $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, $L(x)$ $-$is a family of curves depending on $x = (x_1, x_2, \dots, x_n)$ parameters, and each of the curves of the family entirely belongs to the domain D and connects a pair of points of the boundary of D. Required to find a function $u(\xi)$ inside the domain of D if we know integrals over a family of curves $L(x)$ with a given weight function $p(x, \xi)$:

$$\int_{L(x)} p\,(x,\xi)u(\xi)ds = \vartheta(x). \tag{1}$$

Here $ds-$ is the element of arc length of curve $L(x)$ [9].

This problem was first solved by Radon [8]. For the case when the family of spheres of arbitrary radius whose centers run through the set of points of a fixed hyperplane, the problem (1) is considered in [5], and for the family of spheres of fixed radius in [6]. Various statements of the problem of integral geometry were considered in [7].

In this work we considered reconstruction of the internal structure of the object by their integral data on a familiesof parabolic and polygonal lines. An analytic representation of the solution in the class of smooth finite functions was obtained. Estimates of the solution of the problem in Sobolev spaces are presented, which implies its weak incorrectness. Similar problems were considered in papers [4,2,1]. Uniqueness theorems, stability estimates, and inversion formulas for certain classes of linear problems are obtained in [3].

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2. STABILITY SOLUTION OF FORMULA

We introduce the notation, which we shall use later:

$$(x,y) \in R^2,\ (\xi,\mu) \in R^2,\ \lambda \in R^1,\ \mu \in R^1,$$

$$L_H = \{(x,\, y):\ x \in R^1,\ y \in [0,\, H],\ \ H < \infty\}.$$

In the strip L_H we consider family of polygonal lines, which are defined by relations

$$\Gamma(x,y) = \{(\xi,\eta):\ y-\eta = \ |x-\xi|,\ 0 \le y \le H\}.$$

Problem 1. Recover the function of two variables $u(x,y)$ if in the strip L_H the integrals of it are known from it on the curves of the family $\{\Gamma(x,y)\}$

$$\int\limits_{\Gamma(x,y)} g\,(x,\xi)u(\xi,\eta)d\xi = f(x,y). \tag{2}$$

Theorem 1. *Let the function $f(x,y)$ be known for all $(x,y) \in L_H$, the weight function has the form*

$$g(x,\xi) = \begin{cases} e^{-k(x-\xi)} & if\ \ \xi \le x, \\ 0 & if\ \ \xi > x. \end{cases}$$

Then the solution of equation (2) in the class of continuously differentiable finite functions with support in the strip L_H functions is unique, expressed in terms of the function $f(x,y)$ by the formula

$$u(x,y) = \frac{\partial}{\partial x}f(x,y) + \frac{\partial}{\partial y}f(x,\, y) + k \cdot f(x,\, y) \tag{3}$$

and satisfies the inequalities

$$\|u(x,y)\|_{L_2} \le C\|f(x,y)\|_{W_2^{1,1}}.$$

hereC$-$ is a certain constant.

Proof. Equation (2) can be represented in the following form

$$\int\limits_0^y u\,(x-h,\eta)e^{-k(y-\eta)}d\eta = f(x,y), \tag{4}$$

where $h = y - \eta$.

We apply the Fourier transform to this equation on the first variable:

$$\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{i\lambda x}\, f(x,y)dx = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{i\lambda x}\int\limits_0^y u\,(x-h,\eta)e^{-k(y-\eta)}d\eta dx.$$

We obtain the equation:

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$$\int_0^y \hat{u}\,(\lambda,\eta)e^{-(k-i\lambda)(y-\eta)}d\eta = \hat{f}(\lambda,\, y). \tag{5}$$

We apply the one-sided Fourier transform to the equation (5) on the variable y:

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-i\mu y}\ \hat{f}(\lambda,y)dy = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-i\mu y}\int_0^y \hat{u}\,(\lambda,\eta)e^{-(k-i\lambda)(y-\eta)}d\eta dy.$$

Making the substitution $\tau = y - \eta$, we have

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-i\mu\eta}\ \hat{u}(\lambda,\eta)d\eta\int_0^\infty e^{-(k-i\mu-i\lambda)\tau}\,d\tau = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-i\mu y}\ \hat{f}(\lambda,y)dy \tag{6}$$

Thus, from the equation (6) we obtain:

$$\hat{\hat{u}}(\lambda,\mu)\cdot I(\lambda,\mu) = \hat{\hat{f}}(\lambda,\mu) \tag{7}$$

where

$$I(\lambda,p) = \int_0^\infty e^{-(k-i\lambda-i\mu)t}\,dt = \frac{1}{k-i\lambda-i\mu}. \tag{8}$$

From the equation (7), taking into account (8), we obtain

$$\hat{\hat{u}}(\lambda,\mu) = (k-i\lambda-i\mu)\hat{\hat{f}}(\lambda,\,\mu). \tag{9}$$

We apply the inverse one-sided Fourier transform on μ and the inverse Fourier transform on λ to the equation (8). Taking into account the properties of the Fourier transform, we get the inversion formula

$$u(x,y) = \frac{\partial}{\partial x}f(x,y) + \frac{\partial}{\partial y}f(x,\,y) + k\cdot f(x,\,y).$$

From the last equality we obtain

$$\|u(x,y)\|_{L_2(L_H)} = \left\|\frac{\partial}{\partial x}f(x,y) + \frac{\partial}{\partial y}f(x,\,y) + k\cdot f(x,\,y)\right\|_{L_2(L_H)} \tag{10}$$

Using the Fourier transform properties, we obtain the estimate

$$\|u(x,y)\|_{L_2} \le C\|f(x,y)\|_{W_2^{1,1}}$$

whereC− is a constant. □

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2. The problem of integral geometry on a family of parabolic curve

Problem 2: In the strip $L_H = \{(x,y): x \in R^1, y \in [0,H], H < \infty\}$, reconstruct a function of two variables, $u(x,y)$ if integrals from it are known by the family curves $\Psi(x,y) = \{y-\eta = (x-\xi)^2, 0<\eta<y\}$ with the weight function

$$g(x,\xi) = \begin{cases} x-\xi, & x>\xi \\ \xi - x, & x<\xi \end{cases}$$

are known in the strip L_H

$$\int\limits_{\Psi(x,y)} g(x,\xi)u(\xi,\eta)d\xi = f(x,y). \tag{11}$$

Theorem 2. Let the function be $f(x,y)$ known in the strip L_H. Then the solution of problem 2 in there is a C_0^∞ representation

$$\frac{\partial u_\alpha(x,y)}{\partial x} = \frac{1}{2\alpha\sqrt{\pi^3}}\frac{\partial}{\partial y}\int\limits_0^y\int\limits_{-\infty}^{\infty} e^{\frac{y-\eta-(x-\xi)^2}{4\alpha^2}} \cos\frac{(x-\xi)\sqrt{y-\eta}}{2\alpha^2}\frac{f(\xi,\eta)}{\sqrt{y-\eta}}d\xi d\eta.$$

where $\alpha > 0 -$ the regularization parameter.

Proof. The equation (11) can be written as

$$\int\limits_0^y \left[u(x-h,\eta) - u(x+h,\eta)\right]\frac{d\eta}{2} = f(x,y) \tag{12}$$

Now we apply to both parts of the equation (12) the Fourier transform by variable x:

$$i\int\limits_0^y \hat{u}(\lambda,\eta)\sin(\lambda h)d\eta = \hat{f}(\lambda,y) \tag{13}$$

By differentiating we get

$$\frac{i\lambda}{2}\int\limits_0^y \hat{u}(\lambda,\eta)\frac{\cos\left(\lambda\sqrt{y-\eta}\right)}{\sqrt{y-\eta}}d\eta = \hat{f}_y'(\lambda,y)$$

We act on (13) by the Voltaire operator with the kernel

$$\frac{ch\left(\lambda\sqrt{y-\eta}\right)}{\sqrt{y-\eta}},$$

and apply the Fubini theorem to the left side of the resulting equation:

$$\int\limits_0^y \frac{ch\left(\lambda\sqrt{y-\eta}\right)}{\sqrt{y-\eta}}\hat{f}_\eta'(\lambda,\eta)d\eta = \frac{i\lambda}{2}\int\limits_0^y\int\limits_0^\eta \frac{ch\left(\lambda\sqrt{y-\eta}\right)}{\sqrt{y-\eta}}\hat{u}(\lambda,t)\frac{\cos\left(\lambda\sqrt{\eta-t}\right)}{\sqrt{\eta-t}}dtd\eta =$$

$$= \frac{i\lambda}{2}\int\limits_0^y\left(\int\limits_\eta^y \frac{ch\left(\lambda\sqrt{y-\eta}\right)}{\sqrt{y-\eta}\sqrt{\eta-t}}\cdot\cos\left(\lambda\sqrt{\eta-t}\right)d\eta\right)\hat{u}(\lambda,t)dt$$

We get the equation.

$$\int_0^y \frac{ch\left(\lambda\sqrt{y-\eta}\right)}{\sqrt{y-\eta}} \hat{f}_\eta'(\lambda,\eta)d\eta = \frac{i\lambda}{2}\int_0^y K_1(\lambda,y,t)\hat{u}(\lambda,t)dt$$

where

$$K_1(\lambda,y,t) = 2\int_0^1 \frac{ch\left(\lambda\sqrt{y-t}\sqrt{1-s^2}\right)}{\sqrt{1-s^2}}\cos\left(\lambda s\sqrt{y-t}\right)ds.$$

Having made a replacement $s=\sqrt{\frac{\eta-t}{y-t}}$ and considering the expression for the tabular integral, we get

$$\frac{\pi i\lambda}{2}\int_0^y \hat{u}(\lambda,\eta)d\eta = \int_0^y \frac{ch\left(\lambda\sqrt{y-\eta}\right)}{\sqrt{y-\eta}} \hat{f}_\eta'(\lambda,\eta)d\eta.$$

Having differentiated both parts of the last equation, we come to an expression:

$$i\lambda\hat{u}(\lambda,y) = \frac{2}{\pi}\frac{\partial}{\partial y}\int_0^y \frac{ch\left(\lambda\sqrt{y-\eta}\right)}{\sqrt{y-\eta}} \hat{f}_\eta'(\lambda,\eta)d\eta \qquad (14)$$

Using A.N.Tikhonov's method of regularization of ill-posed problems [10], we introduce a sequence of approximate solutions to the exact solution of the problem in the following way

$$\hat{u}_\alpha(\lambda,y) = i\lambda e^{-\alpha^2\lambda^2}\hat{u}(\lambda,y), \qquad (15)$$

where $\alpha > 0$ -- the regularization parameter. We calculate the inverse Fourier transform from the function $\hat{u}_\alpha(\lambda,y)$

$$F^{-1}[\hat{u}_\alpha(\lambda,y)] = \frac{\partial u_\alpha(x,y)}{\partial x}, \qquad (16)$$

it's a common function, expressed as a convolution.

$$\frac{\partial u_\alpha(x,y)}{\partial x} = \int_{-\infty}^{\infty} W_\alpha(x-\xi)u(\xi,y)d\xi. \qquad (17)$$

Then from (15), (16), (17) and (18) we get

$$\frac{\partial u_\alpha(x,y)}{\partial x} = \frac{1}{2\alpha\sqrt{\pi^3}}\frac{\partial}{\partial y}\int_0^y\int_{-\infty}^{\infty} e^{\frac{y-\eta-(x-\xi)^2}{4\alpha^2}} \cos\frac{(x-\xi)\sqrt{y-\eta}}{2\alpha^2}\frac{f(\xi,\eta)}{\sqrt{y-\eta}}d\xi d\eta. \qquad (18)$$

3. NUMERICAL RECONSTRUCTION ON A FAMILY OF POLYGONAL LINES

3.1. Numerical reconstruction by means of finite difference method

We will investigate the obtained theoretical results by experimental studies. We introduce a uniform grid in the rectangular field $D=[a,\, b]\times[c,\, d]$. We seek approximate solutions of the problem on this rectangle. We obtain the functions $f(x,y)$ from the formula (2). In the arrays of the values of $f(x_i, y_j) = f_{ij}$, we used definitions approximating the values of the function u_{ij}^A and comparing with the analytic values of u_{ij}^E. We rewrite the obtained inversion formula in the form

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$$u(x,y) = \frac{\partial}{\partial x} f(x,y) + \frac{\partial}{\partial y} f(x,y).$$

We find approximate solutions of the problem on the rectangle D. We divide the segment $[a,b]$ on the axis Ox and $[c,d]$ on the axis Oy into $n_x - 1$ and $n_y - 1$ parts respectively: $x_i = a + (i-1)h_x$, $y_j = b + (j-1)h_y$. Approximations of functions $u(x_i, y_j)$ we will denote by u_{ij}^A.

$$u_{ij}^A = \frac{f_{ij+1} - f_{ij-1}}{2h_y} + \frac{f_{i+1j} - f_{i-1j}}{2h_x} \tag{19}$$

Example. A numerical experiment was aimed at comparing the two phantoms of the original and the reconstructed Sheep-Logan phantom. Figure 1 shows the results of restoration.

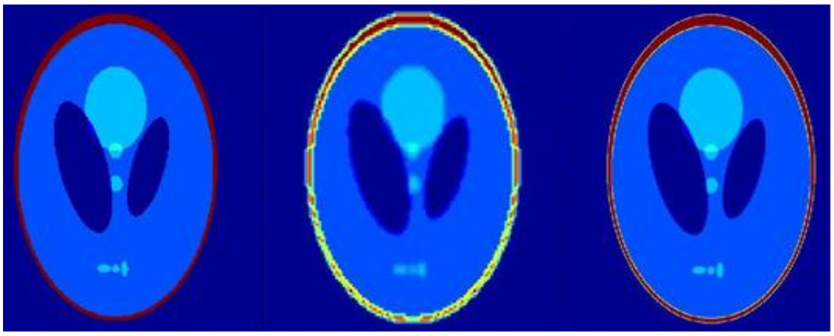

Fig 1. Reconstruction of the test image. a) Original Sheep-Logan phantom, b) Reconstruction by means of finite difference at $\boldsymbol{n_x} = \boldsymbol{n_y}$=128, c) Reconstruction by means of finite difference at $\boldsymbol{n_x} = \boldsymbol{n_y}$=256

3.2. Tikhonov regularization

If the right part in equation (1) is given with noisy data, then the direct use of formula (3) leads to a gross error. Therefore, we use Tikhonov regularization. Suppose that instead of exact right part of $f^E(x,y)$ we are familiar with its approximate value f^δ such that $\|f^\delta - f^E\|_{L_2} \le \delta$, where the δ −upper bound of the right side. We rewrite the obtained inversion formula (9) in the form

$$u(x,y) = \frac{\partial}{\partial x} f^\delta(x,y) + \frac{\partial}{\partial y} f^\delta(x,y). \tag{20}$$

We denote it by $\psi(x,y) = \frac{\partial}{\partial x} f^\delta(x,y)$, $\varphi(x,y) = \frac{\partial}{\partial y} f^\delta(x,y)$. Then formula (20) has the form

$$u(x,y) = \psi(x,y) + \varphi(x,y).$$

The differentiation problem can be written in the form of Volterra equation of the first kind:

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$$\int_a^b K\,(x,s)\psi(s,\cdot)ds = \tilde{f}^{\delta}(x,\cdot) \tag{21}$$

where$\tilde{f}^{\delta}(x,\cdot) = f^{\delta}(x,\cdot) - f^{\delta}(a,\cdot)$

$$\int_c^d K\,(y,\tau)\varphi(\cdot,\tau)d\tau = f^{\delta}(\cdot,y) \tag{22}$$

We rewrite the equation (21) in the form

$$A\psi = \int_a^b K\,(x,s)\psi(s,\cdot)ds = \tilde{f}^{\delta}(x,\cdot).$$

To ensure the stability of the solution of the last equation, we introduce the condition for a minimum of the smoothing functional

$$\Phi_{\alpha}\left[\psi,\tilde{f}^{\delta}\right] = \left\|A\psi - \tilde{f}^{\delta}(x)\right\|_{L_2} + \alpha\|\psi\|_{L_2}, \qquad \alpha > 0. \tag{23}$$

The expansion (23) leads to the following equation of the second kind:

$$\alpha\psi(t,\cdot) + \int_a^b \bar{K}\,(t,s)\psi(s,\cdot)ds = \int_t^b \tilde{f}^{\delta}\,(x,\cdot)dx. \tag{24}$$

where $\bar{K}(t,s) = b - \max\{t,s\}$, $\tilde{f}^{\delta}(x,\cdot) = f^{\delta}(x,\cdot) - f^{\delta}(a,\cdot)$.

We now turn to the numerical solution of the integral equation (24) by the quadrature method. Let the right part of $f(x,y)$be specified in the table on the rectangle $D = \{-a \le x \le b,\ \ c \le y \le d\}$. We divide the segment $[a,b]$ on the axis Ox and $[c,d]$ on the axis Oy into $n_x - 1$ and $n_y - 1$ parts respectively $x_i = a + (i-1)h_x,\ y_j = c + (j-1)h_y,\ \ i = \overline{1,n_x},\ \ j = \overline{1,n_y}$. We discretize the equation (24) on a uniform grid. The integrals in the equation (24) are replaced by integral sums according to the trapezoidal formula. As a result, we obtain systems of linear algebraic equations (SLAE) of the form

$$\alpha\psi_i + \sum_{k=1}^{n_x} \bar{K}\,(t_i,s_k)h_k\psi_k = q_i,\ q_i = \int_{t_i}^b \tilde{f}^{\delta}\,(x,\cdot)dx. \tag{25}$$

$i = \overline{1,n_x},\ h_1 = h_{n_x} = \frac{h_x}{2},\ h_k = h_x,\ k = \overline{2,n_x - 1}$. Let $B-$ matrix with elements $B_{ik} = \bar{K}(t_i,s_k)h_k$. Then the system of the equations (24) with respect to the vector ψ with components $\left(\psi_1,\,\psi_2,\ldots,\psi_{n_x}\right)$ can be written in the form

$$B_{\alpha}\psi \equiv B\psi + \alpha E\psi = q. \tag{26}$$

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where$q -$ vector with components $(q_1/2, q_2, \ldots, q_{n_x})$, and $E -$ unit matrix. Thus, the problem comes to solving SLAE (26). The matrix of this system is symmetric. We describe the procedure for choosing the regularization parameter α. First, a set of values is given α:

$$\alpha_i = \theta \cdot \alpha_{i-1}, \ 0 < \theta < 1, \quad i = 1,2,3, \ldots, m. \tag{27}$$

For a number of values α_x(27) is solved by SLAE (25). We find solutions ψ_{α_i}, with respect to the error of the solution, we obtain:

$$\varepsilon(\alpha_x) = \frac{\left\|\psi_{ij}^{\alpha} - \psi_{ij}^{T}\right\|_{L_2}}{\left\|\psi_{ij}^{T}\right\|_{L_2}} \to \min \tag{28}$$

Then $\psi_{\alpha_{opt\,,x}}$ is found according to the formula (25). Similarly, for the integral equation (26) we use such operations and obtain $\varphi_{\alpha_{opt\,,y}}$.

We will calculate the error value as follows:

$$err = \sqrt{\frac{\sum_{i=2}^{n_x-1}\sum_{j=2}^{n_y-1}|u^A(i,j) - u^E(i,j)|^2}{\sum_{i=2}^{n_x-1}\sum_{j=2}^{n_y-1}|u^E(i,j)|^2}}$$

Example. Let's consider the restoration of the Sheep-Logan phantom.

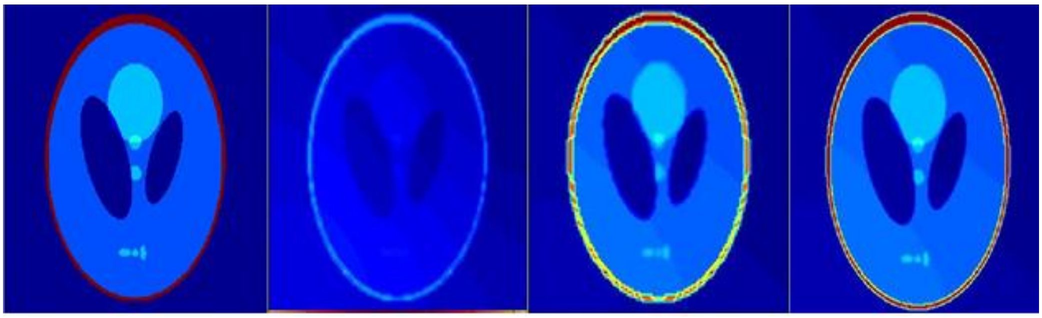

Fig2.Reconstruction of the test image. a) Original Sheep-Logan phantom, b) Reconstruction by means of finite difference at $\boldsymbol{n_x} = \boldsymbol{n_y}$=128, c) Reconstruction by means of Tikhonov regularization at $\boldsymbol{n_x} = \boldsymbol{n_y}$=128,
e) Reconstruction by means of Tikhonov regularization at $\boldsymbol{n_x} = \boldsymbol{n_y}$=512.

Figure 2 shows the restoration of the Sheep-Logan phantom with added 5 percent additive noise. When recovering without noise, with both use and without using regularization, the result is good. One can already see in the figure that a simple restoration is no longer appropriate with regular noise. It can be seen that the method of numerical differentiation gives the image distorted by high-frequency noise, and Tikhonov regularization method gives the image with the sharpest edges.

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Table- 1.Estimation of image recovery by integral data.

$n_x = n_y$	Method		
	RFD without noise	RFD noise – 5%	RTR noise – 5%
128	err=0.3951	err=1.3618	err=0.3989
256	err=0.2961	err=1.7924	err=0.3010
512	err=0.2360	err=2.4558	err=0.2422

where RFD - Reconstruction by means of finite difference, RTR - Reconstruction by means of Tikhonov regularization.

4. CONCLUSION

In this work studied the problem of determining the internal structure of an object by integral data on a family of special curves. An analytic representation of the solution in the class of smooth finite functions is obtained. An estimate of the solutionof the problem in Sobolev space is derived. The obtained theoretical results were confirmed by the experimental data. Tikhonov regularization method was proposed to increase the stability of integrated algorithms for the reconstruction of the tomographicimage, since the application of the finite differences method with sufficiently large noise in the integral data may not give satisfactory results.

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