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Some Properties of Bi-Variate Bi-Periodic Lucas Polynomials

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Abstract
The generalisation of Fibonacci sequence introduced by Edson in 2009. After the generalisation of Fibonacci sequence, Bilgici introduced generalized Lucas sequences. In 2016, Yilmaz and Coskun introduced generalisation of Fibonacci and Lucas polynomials which is known as bi-periodic Fibonacci polynomial and bi-periodic Lucas polynomials. In 2020, Verma and Bala defined bi-variate bi-periodic Fibonacci polynomials. Now, We have defined Bi-variate Bi- periodic Lucas polynomials for $n \geq 2$ with initial conditions $\mathrm{l}_0(x,y) = 2$, $\mathrm{l}_1(x,y) = a_2 x$ by the recurrence relation $\mathrm{l}_n(x,y) = a_1 x \mathrm{l}_{n-1}(x,y) + y \mathrm{l}_{n-2}(x,y)$ if n is even and $\mathrm{l}_n(x,y) = a_2 x \mathrm{l}_{n-1}(x,y) + y \mathrm{l}_{n-2}(x)$ if n is odd. We have obtained generating function for defined polynomial and found n^{th} term of the $\mathrm{l}_n(x,y)$. Investigated relationship between Bi-variate Bi-periodic Fibonacci and Bi-variate Bi- periodic Lucas polynomials. We derived some most popular identities like Cassini's identity, Catalan's identity, d'Ocagne's identity and binomial sum. Convergence of two successive terms of Bi-variate Bi-periodic Lucas polynomial $\mathrm{l}_n(x,y)$ is also discussed.

Keywords: Bi-variate Bi-periodic polynomials, Binet's formula, Cassini's identity, Catalan's identity**,** Generating function

A.M.S. subject classification: 11B83

1. Introduction

Firstly, Yayenie and Edson [6] defined Bi-periodic Fibonacci Sequence (new generalisation of Fibonacci sequence) and found many results. After Yayenie and Edson, many authors gave some important properties and identities of this sequence. Then Bi-periodic Lucas sequence (generalized Lucas sequences) was defined by Bilgici [2] and he gave many results involving Lucas and Fibonacci sequences. Coskun and Yilmaz [13] defined generalized Lucas and Fibonacci polynomials and obtained a few results. After discussion about generalisation of one variable, Catalani [3] introduced generalisation of Bivariate Fibonacci like polynomials. He mainly focused on generalisation of Lucas and Fibonacci polynomials with matrix approach, also some identities and inequalities were obtained by him. After Catalani, many authors studied generalisation of bivariate Fibonacci and Lucas polynomials and introduced many results [1,4,5,7,8,9,10,11].

This Paper is structured in three sections, first section is introductory and in second section, we have defined generalized Bi-variate Bi-periodic Lucas polynomial. Further, in next section, we have obtained generating function for defined polynomial and found n^{th} term of the $\mathrm{l}_n(x,y)$, which is known as Binet's formula. Some important results are also obtained in this section.

2. Definitions and Results

Definition 1. Bi-variate Bi-periodic Lucas polynomial is defined as

$$\mathrm{l}_n(x,y) = \begin{cases} a_1 x \mathrm{l}_{n-1}(x,y) + y \mathrm{l}_{n-2}(x,y) & if\ n\ is\ even \\ a_2 x \mathrm{l}_{n-1}(x,y) + y \mathrm{l}_{n-2}(x) & if\ n\ is\ odd\ . \end{cases} \quad n \geq 2 \qquad (2.1.1)$$

where $\mathrm{l}_0(x,y) = 2,\ \mathrm{l}_1(x,y) = a_2 x$

where $a_1 and\ a_2$ belonging to $\mathbb{R} - \{0\}$

Alternative Definition Bi-variate Bi-periodic Lucas polynomial is defined as

$$\mathrm{l}_n(x,y) = {a_1}^{1-\xi(n)} {a_2}^{\xi(n)} x \mathrm{l}_{n-1}(x,y) + y \mathrm{l}_{n-2}(x,y), \qquad n \geq 2 \qquad (2.1.2)$$

with $\mathrm{l}_0(x,y) = 2,\ \mathrm{l}_1(x,y) = a_2 x$

Parity function $\xi(n)$, can be expressed as

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$$\xi(n)=\begin{cases}0 & if\ n\ is\ even\\ 1 & if\ n\ is\ odd\end{cases}$$

From the definition (2.1.1), characteristic equation of the Bi-variate Bi-periodic Lucas polynomials are

$$\lambda^2-(a_1a_2x^2)\lambda-a_1a_2x^2y=0$$

with roots $\alpha_1(x,y)\ and\ \alpha_2(x,y)$ given by

$$\alpha_1(x,y)\ =\frac{(a_1a_2x^2)+\sqrt{(a_1a_2x^2)^2+4}(a_1a_2x^2)y}{2}\ \ and$$

$$\alpha_2(x,y)=\frac{(a_1a_2x^2)-\sqrt{(a_1a_2x^2)^2+4}(a_1a_2x^2)y}{2}$$

We can simply use $\alpha_1$and α_2 instead of $\alpha_1(x,y)$ and $\alpha_2(x,y)$.

Lemma: If $\{\mathfrak{l}_n(x,y)\}_{n=0}^{\infty}$ is defined by (2.1.1) then:

$$\mathfrak{l}_{2n}(x,y)=\big((a_1a_2x^2)+2y\big)\mathfrak{l}_{2n-2}(x,y)-y^2\mathfrak{l}_{2n-4}(x,y) \quad (2.1.3)$$

$$\mathfrak{l}_{2n+1}(x,y)=\big((a_1a_2x^2)+2y\big)\mathfrak{l}_{2n-1}(x,y)-y^2\mathfrak{l}_{2n-3}(x,y) \quad (2.1.4)$$

Proof: This can be solved easily with the help of definition (2.1.1)

$$\mathfrak{l}_{2n}(x,y)\ =\ a_1x\mathfrak{l}_{2n-1}(x,y)+y\mathfrak{l}_{2n-2}(x,y) \quad (2.1.5)$$

After substituting the value of $\mathfrak{l}_{2n-1}(x,y)$ in equation (2.1.5), we will get

$$=a_1x[a_2x\mathfrak{l}_{2n-2}(x,y)+y\mathfrak{l}_{2n-3}(x,y)]+y\mathfrak{l}_{2n-2}(x,y)$$

$$=(a_1a_2x^2+y)\mathfrak{l}_{2n-2}(x,y)+y[a_1x\mathfrak{l}_{2n-3}(x,y)]$$

$$=(a_1a_2x^2+y)\mathfrak{l}_{2n-2}(x,y)+y\ [\mathfrak{l}_{2n-2}(x,y)-y\mathfrak{l}_{2n-4}(x,y)]$$

$$=(a_1a_2x^2+y)\mathfrak{l}_{2n-2}(x,y)-y^2\mathfrak{l}_{2n-4}(x,y)$$

Similarly, by simple steps of calculation as performed above, we can conclude the following results for odd indices

$$\mathfrak{l}_{2n+1}(x,y)=\big((a_1a_2x^2)+2y\big)\mathfrak{l}_{2n-1}(x,y)-y^2\mathfrak{l}_{2n-3}(x,y)$$

3. Generating function and Binet's formula of $\mathfrak{l}_n(x,y)$, and important results

Theorem (Generating Function of $\mathfrak{l}_n(x,y)$): The generating function of the Bivariate Bi-periodic Lucas Polynomials $\{\mathfrak{l}_n(x,y)\}_{n=0}^{\infty}$ is given by

$$£(t)=\sum_{n=0}^{\infty}\mathfrak{l}_n(x,y)t^n$$

$$£(t)=\frac{a_2xt+2-[(a_1a_2x^2)+2y]t^2+[a_2xy]t^3}{1-\big((a_1a_2x^2+2y)\big)t^2+y^2t^4}$$

Proof: We define

$$£_0(t)=\sum_{n=0}^{\infty}\mathfrak{l}_{2n}(x,y)t^{2n}$$

and

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$$£_1(t) = \sum_{n=0}^{\infty} \mathrm{l}_{2n+1}(x,y)t^{2n+1}$$

So that

$$£(t) = £_0(t) + £_1(t)$$

We have

$$£_0(t) = \sum_{n=0}^{\infty} \mathrm{l}_{2n}(x,y)t^{2n}$$

$$£_0(t) = 2 + \left((a_1a_2x^2) + 2y\right)t^2 + \sum_{n=2}^{\infty} \mathrm{l}_{2n}(x,y)t^{2n}$$

Replace the value of $\mathrm{l}_{2n}(x,y)$ from equation (2.1.3) and we get,

$$£_0(t) = 2 + \left((a_1a_2x^2) + 2y\right)t^2$$
$$+ \sum_{n=2}^{\infty} \left((a_1a_2x^2) + 2y\right)\mathrm{l}_{2n-2}(x,y) - y^2\mathrm{l}_{2n-4}(x,y)t^{2n}$$

$$£_0(t) = 2 + \left((a_1a_2x^2) + 2y\right)t^2$$
$$+\left((a_1a_2x^2) + 2y\right)t^2 \sum_{n=2}^{\infty} \mathrm{l}_{2n-2}(x,y)t^{2n-2} - y^2t^4\,£_0(t)$$

$$£_0(t) = 2 + \left((a_1a_2x^2) + 2y\right)t^2$$
$$+\left((a_1a_2x^2) + 2y\right)t^2 \sum_{n=2}^{\infty} \{\mathrm{l}_{2n-2}(x,y)t^{2n-2} + 2 - 2\} - y^2t^4\,£_0(t)$$

$$£_0(t) = 2 + \left((a_1a_2x^2) + 2y\right)t^2 \; - 2\left((a_1a_2x^2) + 2y\right)t^2$$
$$+\left((a_1a_2x^2) + 2y\right)t^2£_0(t) - -y^2t^4£_0(t)$$

Solving further, we get

$$£_0(t) = \frac{2 - [(a_1a_2x^2) + 2y]t^2}{1 - \left((a_1a_2x^2 + 2y)\right)t^2 + y^2t^4}$$

Similarly, we find

$$£_1(t) = \sum_{n=0}^{\infty} \mathrm{l}_{2n+1}(x,y)t^{2n+1}$$

Substitute the value of $\mathrm{l}_{2n+1}(x,y)$ from equation (2.1.4), we have

$$£_1(t) = a_2xt + a_2x\left((a_1a_2x^2) + 3y\right)t^3 + \sum_{n=2}^{\infty} \mathrm{l}_{2n+1}(x,y)t^{2n+1}$$

$$£_1(t) = a_2xt + a_2x\left((a_1a_2x^2) + 3y\right)t^3 + \sum_{n=2}^{\infty} \mathrm{l}_{2n+1}(x,y)t^{2n+1}$$

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$$£_1(t) = a_2xt + a_2x\big((a_1a_2x^2) + 3y\big)t^3$$
$$+\big((a_1a_2x^2) + 2y\big)t^2 \sum_{n=2}^{\infty} \mathfrak{l}_{2n-1}(x,y)t^{2n-1} - y^2t^4\, £_0(t)$$
$$£_1(t) = a_2xt + a_2x\big((a_1a_2x^2) + 3y\big)t^3$$
$$+\big((a_1a_2x^2) + 2y\big)t^2 \sum_{n=2}^{\infty} \{\mathfrak{l}_{2n-1}(x,y)t^{2n-1} + a_2xt - a_2xt\} - y^2t^4\, £_0(t)$$
$$£_1(t) = a_2xt + a_2x\big((a_1a_2x^2) + 3y\big)t^3 \;\; - a_2x\big((a_1a_2x^2) + 2y\big)t^3$$
$$+\big((a_1a_2x^2) + 2y\big)t^2 £_0(t) - \; -y^2t^4 £_0(t)$$

Solving further, we obtain

$$£_1(t) = \frac{a_2xt + [a_2xy]t^3}{1 - \big((a_1a_2x^2 + 2y)\big)t^2 + y^2t^4}$$

We know,

$$£(t) = £_0(t) + £_1(t)$$

We get

$$£(t) = \frac{a_2xt + 2 - [(a_1a_2x^2) + 2y]t^2 + [a_2xy]t^3}{1 - \big((a_1a_2x^2 + 2y)\big)t^2 + y^2t^4}$$

Theorem (Binet's Formula) The n^{th} term of the Bi-variate Bi-periodic Lucas polynomial $\mathfrak{l}_{\boldsymbol{n}}(\boldsymbol{x}, \boldsymbol{y})$ is given by

$$\mathfrak{l}_{\boldsymbol{n}}(\boldsymbol{x}, \boldsymbol{y}) = \frac{(a_2x)^{\xi(n)}\{(\alpha_1)^n + (\alpha_2)^n\}}{(a_1a_2x^2)^{\left\lfloor \frac{n+1}{2} \right\rfloor}}$$

Where α_1 and α_2 are roots of the characteristic equation

$$\lambda^2 - (a_1a_2x^2)\lambda - a_1a_2x^2y = 0$$

Proof: Firstly, note that α_1, α_2 and their following properties will be used throughout the proof.

(i) $\alpha_1 + \alpha_2 = a_1a_2x^2$

(ii) $\alpha_1\alpha_2 = -a_1a_2x^2y$

(iii) $(\alpha_1 + y)(\alpha_2 + y) = y^2$

(iv) $(\alpha_1 + y) = \frac{\alpha_1{}^2}{a_1a_2x^2}$

(v) $(\alpha_2 + y) = \frac{\alpha_2{}^2}{a_1a_2x^2}$

(vi) $-\alpha_2(\alpha_1 + y) = y\alpha_1$

(vii) $-\alpha_1(\alpha_2 + y) = y\alpha_2$

Since $\frac{\alpha_1+y}{y^2}$ and $\frac{\alpha_2+y}{y^2}$ are roots of

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$$1-\left((a_1a_2x^2+2y)\right)t^2+y^2t^4=0$$

If we assume

$$£_0(t)=\sum_{n=0}^{\infty}\mathfrak{l}_{2n}(x,y)\,t^{2n}$$

and

$$£_1(t)=\sum_{n=0}^{\infty}\mathfrak{l}_{2n+1}(x,y)\,t^{2n+1}$$

Then

$$£(t)=£_0(t)+£_1(t)$$

By using Maclaurin's Series expansion

$$\frac{A+BZ}{Z^2-C}=\sum_{n=0}^{\infty}AC^{-n-1}Z^{2n}-\sum_{n=0}^{\infty}BC^{-n-1}Z^{2n+1}$$

and above-mentioned identities, we simplify both $£_0(t)$ and $£_1(t)$ as follows:

$$£_0(t)=\frac{1}{(\alpha_1-\alpha_2)}.\sum_{n=0}^{\infty}\left(\frac{a_1a_2x^2y+[(a_1a_2x^2)+2y]\alpha_1}{(a_1a_2x^2)^{n+1}y^2}(\alpha_2)^{2n+2}\right.$$
$$\left.-\frac{a_1a_2x^2y+[(a_1a_2x^2)+2y]\alpha_2}{(a_1a_2x^2)^{n+1}y^2}(\alpha_1)^{2n+2}\right)t^{2n}$$

Further solve and use this equation $[(a_1a_2x^2)+2y]\alpha_1-(\alpha_1)^2$=y$(\alpha_1-\alpha_2)$ in above equation .Similarly $[(a_1a_2x^2)+2y]\alpha_2-(\alpha_2)^2$=y$(\alpha_2-\alpha_1)$ and we get

$$£_0(t)=\sum_{n=0}^{\infty}\frac{\{(\alpha_1)^{2n}+(\alpha_2)^{2n}\}}{(a_1a_2x^2)^n}t^{2n}$$

We solve $£_1(t)$ with the same approach used in $£_0(t)$ and we get the value of

$$£_1(t)=\sum_{n=0}^{\infty}\frac{\{(\alpha_1)^{2n+1}+(\alpha_2)^{2n+1}\}}{(a_1a_2x^2)^{n+1}}t^{2n+1}$$

We know that

$$£(t)=£_0(t)+£_1(t)$$

So we find $£(t)=\sum_{n=0}^{\infty}\frac{(a_2x)^{\xi(n)}\{(\alpha_1)^n+(\alpha_2)^n\}}{(a_1a_2x^2)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}t^{2n}$

Thus

$$\mathfrak{l}_n(x,y)\;=\frac{(a_2x)^{\xi(n)}\{(\alpha_1)^n+(\alpha_2)^n\}}{(a_1a_2x^2)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}$$

Theorem 1: For any two non-zero real numbers. The polynomial sequence $\{\mathfrak{l}_n(x,y)\}_{n=0}^{\infty}$satisfy Cassini's identity such that:

a) Cassini's Identity:

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$$\left(\frac{a_1}{a_2}\right)^{\xi(n+1)} \mathfrak{l}_{n-1}(x,y)\mathfrak{l}_{n+1}(x,y) - \left(\frac{a_1}{a_2}\right)^{\xi(n)} (\mathfrak{l}_n(x,y))^2 = (-1)^{n+1}\{(a_1a_2x^2) + 4y\}$$

b) Catalan identity**:**

$$\left(\frac{a_1}{a_2}\right)^{\xi(n+r)} \mathfrak{l}_{n-r}(x,y)\mathfrak{l}_{n+r}(x,y) - \left(\frac{a_1}{a_2}\right)^{\xi(n)} (\mathfrak{l}_n(x,y))^2 = (-1)^{n+r}\{\mathfrak{l}_{2r}(x,y) - 2(-y)^r$$

c) Binomial Sum:

$$\sum_{j=0}^{n} \binom{n}{j} (a_2x)^{\xi(j+1)}(a_1a_2x^2)^{\left|\frac{j+1}{2}\right|} \mathfrak{l}_j(x,y) = a_2x\mathfrak{l}_{2n}(x,y)$$

d) d'Ocagne'sIdentity:

$$(a_2x)^{\xi(mn+m)}(a_1x)^{\xi(mn+n)}\mathfrak{l}_{m+1}(x,y)\mathfrak{l}_n(x,y) - (a_2x)^{\xi(mn+n)}((a_1x))^{\xi(mn+m)}\mathfrak{l}_m(x,y)\mathfrak{l}_{n+1}(x,y)$$
$$= \big(-(y)\big)^n[(a_1x)(a_2x) + 4y]\mathcal{B}_{m-n}(x,y).$$

Where $\mathcal{B}_n(x,y) = \frac{(px)^{1-\xi(n)}}{(prx^2)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\beta_1^{\lfloor\frac{n}{2}\rfloor}(\beta_1+(s-q)y)^{n-\lfloor\frac{n}{2}\rfloor}-\beta_2^{\lfloor\frac{n}{2}\rfloor}(\beta_2+(s-q)y)^{n-\lfloor\frac{n}{2}\rfloor}}{\beta_1-\beta_2}\right)$ is binet's formula for bivariate bi-periodic Fibonacci polynomials[13]. If we take $q = s = 1$ and $= a_2$, $q = a_1$. Then $\mathcal{B}_{m-n}(x,y) = \frac{(a_2x)^{1-\xi(m-n)}}{(a_2a_1x^2)^{\lceil\frac{m-n}{2}\rceil}}\left(\frac{{\beta_1}^{m-n}-{\beta_2}^{m-n}}{\beta_1-\beta_2}\right)$. Where characteristic equation for bivariate bi-periodic Fibonacci polynomials and bivariate bi-periodic Lucas polynomials are equal if above conditions are applied.

Theorem 2: For any two successive terms of Bi-variate Bi-periodic Lucas polynomial $\mathfrak{l}_n(x,y)$, The following results will be satisfied

$\lim_{n\to\infty} \frac{\mathfrak{l}_{2n+1}(x,y)}{\mathfrak{l}_{2n}(x,y)} = \frac{\alpha_1(x,y)}{a_1x}$ and $\lim_{n\to\infty} \frac{\mathfrak{l}_{2n}(x,y)}{\mathfrak{l}_{2n-1}(x,y)} = \frac{\alpha_1(x,y)}{a_2x}$.

Proof*:* We solve above theorem with the help of binet's formula and assume that

$|\alpha_2(x,y)| \leq \alpha_1(x,y)$ and $\lim_{n\to\infty} \left(\frac{\alpha_2(x,y)}{\alpha_1(x,y)}\right)^n = 0$,

$$\lim_{n\to\infty} \frac{\mathfrak{l}_{2n+1}(x,y)}{\mathfrak{l}_{2n}(x,y)} = \lim_{n\to\infty} \frac{\frac{\{(\alpha_1(x,y))^{2n+1} + (\alpha_2(x,y))^{2n+1}\}}{(a_1a_2x^2)^{n+1}}}{\frac{\{(\alpha_1(x,y))^{2n} + (\alpha_2(x,y))^{2n}\}}{(a_1a_2x^2)^n}}$$

$$= \lim_{n\to\infty} \frac{{\alpha_1}^{2n+1}(x,y)\left(1 - \frac{{\alpha_2}^{2n+1}(x,y)}{{\alpha_1}^{2n+1}(x,y)}\right)}{(a_1x){\alpha_1}^{2n}(x,y)\left(1 - \frac{{\alpha_2}^{2n}(x,y)}{{\alpha_1}^{2n}(x,y)}\right)}$$

$$= \lim_{n\to\infty} \frac{\alpha_1(x,y)}{(a_1x)}$$

$$= \frac{\alpha_1(x,y)}{(a_1x)}$$

Similarly, we will get $\lim_{n\to\infty} \frac{\mathfrak{l}_{2n}(x,y)}{\mathfrak{l}_{2n-1}(x,y)} = \frac{\alpha_1(x,y)}{a_2x}$.

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