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Formation of Triangles with Rational and Integral Sides in the Paisacika Mathematics of *Ganita Kaumudi*

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ABSTRACT. Methods for the formation of triangles with rational and integral sides, with the sides differing from the base by unity, as proposed by Nārāyaṇa Paṇḍita in his *Gaṇita Kaumudī* (1356 AD)[1] are described in detail and the relevant formulas are derived, using indeterminate equations and recursive techniques.

1.INTRODUCTION

Indian mathematics evolved right from the Vedic times and had achieved great heights through the ages in the field of arithmetic, algebra, geometry, combinatorics, astronomy etc. For example, in the field of indeterminate equations, highly efficient methods were developed right from the days of Brahmagupta. Equations of the first and second degrees of the type $ax - by = c$ and $Nx^2 + 1 = y^2$ were discussed at length by Aryabhaṭa(5th century AD) and Brahmagupta (7th Century AD)[2].

This paper is concerned with the development of such rational triangles expounded in the rules of Gaṇita Kaumudi, a book on mathematics composed by the scholar Narayaṇa Paṇḍita in the year 1356AD. An English translation with explanatory notes is also available[3]. In the chapters on geometry, there is a separate portion called *Paisachikam* meaning devilishly difficult mathematics. In this section, Narayaṇa explains the procedure for formation of triangles with rational sides and has also stated recursive formulae for the development of triangles with integral sides. Practical examples have also been given alongside for the student to solve.

2. METHODS

2.1 Formation of rational triangles with sides differing by unity from the base

The rule attempts to create a triangle with base x and the sides $x + 1$ and $x - 1$ where x can be integer or fraction. The perpendicular from the vertex to the base which is also evaluated in the process is also integer or fraction. The relevant rule is as follows:

Rule 118:

"Divide twice an optional number by the square of the optional number less 3. Add 1 to thrice the square of the quotient.Twice the square root of the sum is the base. 1 added to and subtracted from the base are the flank sides.

The explanation of the rule is:

1. Choose an optional number m.
2. Twice the number = 2m

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3. This divided by the square of the optional number less 3 gives $2m/(m^2 - 3)$.
4. Adding 1 to thrice the square of the result we get $3(2m/(m^2-3))^2 + 1$
5. Base equals twice the square root of the number in step 4 i.e. base = $2\sqrt{3(\frac{2m}{m^2-3})^2+1}$

$= 2\sqrt{\frac{12m^2}{(m^2-3)^2}+1} = 2\sqrt{\frac{(m^2+3)^2}{(m^2-3)^2}} = 2\frac{m^2+3}{m^2-3}$

i.e. $x = 2\frac{m^2+3}{m^2-3}$

The flank sides are x – 1 and x + 1.

Proof:

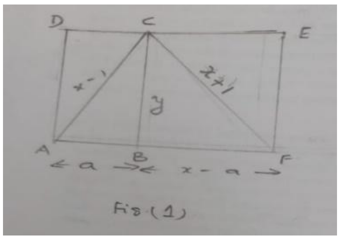

Fig 1. Rational triangle

The rational triangle is created from a combination of two rectangles with diagonals x – 1 and x + 1 as shown in Fig. 1.
AC = diagonal of smaller rectangle = x – 1.
CF = diagonal of larger rectangle = x + 1.
AF = base = x.
CB = perpendicular from vertex C to base = y.
Now base AF = AB + BF
Let AB = a ; Then BF = x – a
From the triangles CBF and CBA, we have

$$(x+1)^2-(x-a)^2=y^2 \quad (1)$$
$$(x-1)^2-a^2=y^2 \quad (2)$$

(1) – (2) gives,

$$(x+1)^2-(x-a)^2-(x-1)^2+a^2=0 \quad (3)$$

This yields,

$$4x-x^2+2ax=0 \quad (4)$$

Cancelling x throughout,

$$4-x+2a=0 \quad (5)$$
$$a=\frac{x-4}{2} \quad (6)$$
$$BF=x-a=x-(\frac{x}{2}-2)=\frac{x}{2}+2 \quad (7)$$

Now the length of perpendicular y is given by,

$$y^2=AC^2-AB^2=(x-1)^2-a^2=(x-1)^2-(\frac{x}{2}-2)^2 \quad (8)$$

Simplifying,

$$y^2=\frac{3x^2}{4}-3 \quad (9)$$

Thus we have the indeterminate equation or what is known as *Varga Prakṛti* namely,

$$\frac{3x^2}{4}-3=y^2 \quad (10)$$

This can be solved provided we reduce it to the form,

$$Nx^2+1=y^2 \quad (11)$$

The rational solution of the equation has been given in Page 78 of Bija Pallava[3] of Kṛṣṇa Daivajna by Dr. Sita Sundar Ram.
We now consider the identity

$$N(2x_1)^2+(N-{x_1}^2)^2=(N+{x_1}^2)^2 \quad (12)$$

This is based on the relation

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$4ab + (a-b)^2 = (a+b)^2.$ (13)

Dividing Eqn (12) throughout by $(N - {x_1}^2)^2$,

$N \times (\frac{2x_1}{N-{x_1}^2})^2 + 1 = (\frac{N+{x_1}^2}{N+{x_1}^2})^2$ (14)

assuming $that\ N\ \#\ {x_1}^2$ since N is not a square number.

Comparing Eqn. (14) with Eqn. (11) we get the rational solution as,

$x = \frac{2x_1}{|N-{x_1}^2|}$ (15)

and

$y = \frac{N+{x_1}^2}{|N-{x_1}^2|}$ (16)

x and y are called the *Kaniṣṭha* and *Jyeṣṭha* respectively.

Reverting to Eqn. (10),

Let x = 2t. The equation becomes

$3t^2 - 3 = y^2$ (17)

In order to reduce the constant term (called kṣepa) to 1, we assume y = 3p.

Eqn. (12) reduces to,

$3t^2 - 3 = (3p)^2 = 9p^2$ (18)

$t^2 - 1 = 3p^2$

Or

$3p^2 + 1 = t^2$ (19)

Eqn. (19) is of the form of Eqn. (11). Therefore the general solution is,

$p = \frac{2x_1}{|N-{x_1}^2|} = \frac{2x_1}{{x_1}^2-3}$(20)

And

$t = \frac{N+{x_1}^2}{|N-{x_1}^2|} = \frac{{x_1}^2+3}{{x_1}^2-3}$ (21)

We can replace x_1 with m and get the results as

$p = \frac{2m}{m^2-3}$ (22)

and

$t = \frac{\mathrm{m}^2+3}{m^2-3}$ (23)

The above solution of the equation of the type (19) has been stated by Narayaṇa himself in the following verse (No. 75 and 76), in his work *Bijagaṇitavatamsa* in page no. 153.

Rule Number 75

"Twice an optional number divided by the difference between the square of that optional number and the *Prakṛti* (N) happens to be the lesser root*Kaniṣṭha* (of the equation of a square nature when unity is the additive). From that, with unity as additive, the greater root should be obtained. Thus,

$p = \frac{2m}{m^2-N} = \frac{2m}{m^2-3}$ (24)

$t^2 = 3p^2 + 1 = 3(\frac{2m}{m^2-3})^2 + 1 = \frac{12m^2}{(m^2-3)^2} + 1 = (\frac{m^2+3}{m^2-3})^2$ (25)

Therefore,

$t = \frac{m^2+3}{m^2-3}$ (26)

We thus have

$y = 3p = 3\frac{2m}{m^2-3} = \frac{6m}{m^2-3}$ (27)

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and

$$x = 2t = 2(\frac{m^2+3}{m^2-3}) \quad (28)$$

Based on the above result, Narayaṇa gives the student a problem asking him to tell him the triangles in which the sides increase by unity. The results given are

(1) 3, 4 and 5
(2) 13, 14 and 15
(3) 38/13, 25/13 and 51/13
(4) 26/11, 15/11 and 37/11

These are obtained by taking m = 1,2,4 and ½ and substituting in the expression for x.

2.2 Formation of integral triangles under given conditions

While the procedure described above gives rise to rational solutions for the sides of a triangle, we observe from the results of the example given by Narayaṇa, the sides are integers in some cases and fractions in others. Narayaṇa proceeds to give a procedure for evaluating the sides of a triangle with unit difference with the base but in integers only. For this, he first proposes a basic right-angled triangle of sides 3, 4 and 5 and from these he proceeds to derive a recursive formula for obtaining infinite number of triangles with integer sides, the sides in each triangle having unit difference with the base. The relevant rules stated by him are as follows:

Rule 119 and 120.
The meaning of the rules is as follows. The perpendicular is 3 units and the base is 4 units of the first right-angled triangle. Infinite pairs of right-angled triangles are produced in which the sides increase by unity. In these, 3 times the base of the previous triangle added to the still previous perpendicular is the perpendicular from the vertex to the base. The base of the triangle is twice the sum of the previous perpendicular added to the previous base. The triangles set in opposition are right-angled and in all such triangles, 1 added to and subtracted from the base are the flank sides.
The rule gives a method of finding integral triangles in which each side and the base differ by unity and each such triangle is divided by the perpendicular from the vertex into a pair of integral right-angled triangles.
Explanation of the rules are as follows. Let $b_{n,}$ p_n refer to the base and perpendicular of the nth triangle generated. Then,

$$p_n = 3b_{n-1} + p_{n-2} \quad (29)$$

and

$$b_n = 2(b_{n=1} + p_{n-1}) \quad (30)$$

This recursive rule can be used for the creation of integral triangles in which each side and the base differ by unity. Each such triangle is divided by the perpendicular from the vertex into a pair of integral right-angled triangles.

Proof:
Let $b_{1,}b_{2,}b_3 \ldots b_n$ be the bases and $p_{1,}p_{2,}p_3 \ldots p_n$ be the perpendiculars from the vertex of the series of triangles formed. Let $a_{1,}a_{2,}a_3 \ldots a_n$ and $c_{1,}c_{2,}c_3 \ldots c_n$ be the flank sides of these triangles. Thus, according to the rule, Eqn (29) and Eqn. (30) follow. Also,

$$a_n = b_n + 1 \quad (31)$$

and

$$c_n = b_n - 1 \quad (32)$$

We have already shown in the previous section that the equation for such a triangle with sides differing by unity is

$$\frac{3x^2}{4} - 3 = y^2 \quad (33)$$

where x is the base and y is the perpendicular. One solution is x=2, y=0 which is trivial as perpendicular cannot be zero. Now, consider the equation

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$$\frac{3x^2}{4} + 1 = y^2 \quad (34)$$

with unity as additive. Now, successive solutions of Eqn. (33) can be got by a procedure called repeated samaasa bhavana which is described in detail vide page no. 145 of *Bijagaṇitavatamsa* [5] as follows:

Assume b_1, p_1 are the first such roots satisfying

$$\frac{3{b_1}^2}{4} - 3 = {p_1}^2 \quad (35)$$

We find x = 2 and y = 2 satisfy the Eqn. (34). Write the values of x, y and additive or constant one below the other for the Eqn. (33) and Eqn. (34). Also, set down the value of the coefficient of x^2 adjoining to the left of the roots of Eqn. (33). Now we get the following configuration,

	x	y	additive
¾	b_1	p_1	-3
	2	2	1

The new roots b_2 and p_2 of the Eqn. (33) are obtained as follows:

Cross-multiply b_1 with 2 and p_1 with 2 and this becomes equal to b_2 i.e.

$$b_2 = 2b_1 + 2p_1 \quad (36)$$

For getting p_2, multiply ¾ with the product $2b_1$ and then add $2p_1$ to the same. Thus

$$p_2 = (3/4) * 2b_1 + 2p_1 = (3/2)b_1 + 2p_1 \quad (37)$$

Now, we execute a second *bhavana* replacing b_1,p_1 with b_2,p_2.

	x	y	additive
¾	b_2	p_2	-3
	2	2	1

Now,

$$b_3 = 2b_2 + 2p_2 \quad (38)$$

and

$$p_3 = \frac{3}{2}b_2 + 2p_2 \quad (39)$$

and so on. Thus, we have

$$b_{n-1} = 2b_{n-2} + 2p_{n-2} \quad (40)$$

and

$$p_{n-1} = \frac{3}{2}b_{n-2} + 2p_{n-2} \quad (41)$$

Again,

$$b_n = 2b_{n-1} + 2p_{n-1} \quad (42)$$

and

$$p_n = \frac{3}{2}b_{n-1} + 2p_{n-1} \quad (43)$$

Or

$$p_n = 2(\frac{3}{2}b_{n-2} + 2p_{n-2}) + \frac{3}{2}b_{n-1} \quad (44)$$

Or

$$p_n = p_{n-2} + (3p_{n-2} + 3b_{n-2}) + \frac{3}{2}b_{n-1} \quad (45)$$

Further simplifying

$$p_n = p_{n-2} + \frac{3}{2}(2p_{n-2} + 2b_{n-2}) + \frac{3}{2}b_{n-1} \quad (46)$$

Substituting Eqn. (40) into Eqn. (46),

$$p_n = p_{n-2} + \frac{3}{2}b_{n-1} + \frac{3}{2}b_{n-1} \quad (47)$$

$$p_n = p_{n-2} + 3b_{n-1} \quad (48)$$

Thus, we have two recursive rules Eqn. (42) and Eqn. (48).

3 EXAMPLES

Narayaṇa himself has illustrated through figures, how a number of triangles can be generated using Eqn. (42) and Eqn. (48). The first or basic triangle is with base 4 and perpendicular 3. As per the formula, $b_1=4$, $c_1=3$ and $a_1 = 5$. Using $b_2=2b_1+2p_1=8+6=14$ and $p_2=(3/2)b_1+2p_1=12$ the flank sides are (14 + 1) and (14 – 1) i.e. 15 and 13. Thus we have a triangle of base 14, flank sides 13 and 15 and perpendicular 12. The other triangles using the recursive formula generated are:

1. Base = 52, flank sides = 51, 53 and perpendicular = 45
2. Base = 194, Flank sides = 193, 195 and perpendicular = 168.
3. Base = 724, Flank sides = 723, 725 and perpendicular = 627.
4. Base = 2702, Flank sides = 2701, 2703 and perpendicular = 2340

4. CONCLUSION

Two different methods for formation of scalene triangles with sides differing by unity from the base and with rational or integral sides and perpendiculars as stated by Narayaṇa Paṇḍita have been explained using solutions of indeterminate equations and through recursive methods.

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