Annals of R.S.C.B., ISSN:1583-6258, Vol. 25, Issue 4, 2021, Pages. 5775 - 5783
Received 05 March 2021; Accepted 01 April 2021.

A Parametric Nonlinear Programming Approach to Analyzing Cyclic Queueing Networks in Fuzzy Environments

H. Merlyn Margaret[a] and P. Thirunavukarasu[b]

[a]*Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, SRM Nagar, Kattankulathur – 603203, Kanchipuram, Chennai, TN, India, merlynh@srmist.edu.in*
[b]*P.G. and Research Department of Mathematics, Periyar E.V.R. College (Autonomous), Tiruchirappalli – 620023, TN, India, ptavinash1967@gmail.com*

Abstract

Queueing theory has a wide range of applications in engineering and in the sciences. Jackson networks are a widely studied class of queueing systems, with applications in models of machine repair, communication and in computer networks. Real world data is imprecise, and thus there is an intrinsic fuzziness associated with the data. This fuzziness is resolved through the use of fuzzy theoretic techniques. In this paper, we study cyclic queueing systems, a special class of Jackson networks, in fuzzy environments, wherein the data is intrinsically imprecise. We propose a solution procedure that enables one to arrive at the fuzzified performance measures of such systems. The analysis is effectively reduced to that of an optimization problem, which lies under the purview of parametric nonlinear programming. We also use the Yager ranking index to arrive at the equivalent crisp performance measures. A numerical example is solved to illustrate the solution procedure.

Keywords: Jackson networks, cyclic queues, parametric nonlinear programming, fuzzy sets;

1. Introduction

The theory of fuzzy sets and logic was first conceived of by Lotfi A. Zadeh in 1965 [7]. He extended the concepts of classical set theory to account for uncertainty and vagueness in data and laid the foundations of modern fuzzy set theory. A considerable and significant development of the subject was due to Zadeh himself. This theory is now applied to a wide range of scientific areas to model uncertainty in data.

Other significant researchers in this area include Dubois and Prade [16], Kaufmann [20], Mizumoto and Tanaka [17], Nakamura [19] etc. Applications include areas like control systems, statistics, pattern classification, neural networks, communication, queueing systems and so on.

Our focus is on using the methods of fuzzy set theory to analyze queueing systems. Queueing theory studies queues from a probabilistic point of view. Service times and arrival times are assumed to follow probability distributions. In the fuzzy analysis of such systems, the intrinsic fuzziness in the parameters of these probability distributions are taken into account. Notable papers in this area include those by Li and Lee [6], Negi and Lee [5] and Buckley et al. [4]. Various techniques have been employed to analyze queueing systems in fuzzy environments, including the traditional α-cut arithmetic, the Dong-Shah-Wong algorithm [14], LR arithmetic [15] and parametric nonlinear programming. Kao et al. and Chen et al. use parametric nonlinear programming techniques to analyze fuzzy queues in [11] and [1]. Mukeba et al. in [18] use LR arithmetic in their analysis of the fuzzy M/M/1 queue.

In this paper, we analyze a special class of queueing systems called cyclic queueing systems, in fuzzy environments. We effectively reduce the analysis to that of solving a pair of parametric nonlinear programs. Finally, we use the Yager ranking index to defuzzify the fuzzy output.

The overview of this paper is as follows. Sec. 2 discusses necessary preliminaries. Sec. 3 defines trapezoidal fuzzy numbers. Sec. 4 describes the queueing model in discussion – cyclic queues, in brief. Sec. 5 describes our solution procedure, and Sec. 6 validates the solution procedure by means of an example. Sec. 7 concludes the study.

2. Preliminaries

2.1. Fuzzy set-theoretic definitions

A *fuzzy set* [2] $\widetilde{\boldsymbol{A}}$ is an ordered pair $(\boldsymbol{X}, \boldsymbol{A})$, consisting of a set $\boldsymbol{X}$ that we shall refer to as the *universe* (or the *domain of discourse*), and a function $\boldsymbol{A} : \boldsymbol{X} \to [\mathbf{0}, \mathbf{1}]$, called its *membershipfunction*, that maps the universe into the interval

Annals of R.S.C.B., ISSN:1583-6258, Vol. 25, Issue 4, 2021, Pages. 5775 - 5783
Received 05 March 2021; Accepted 01 April 2021.

$[\mathbf{0},\mathbf{1}]$. This map induces a measure of membership in the fuzzy set $\widetilde{\boldsymbol{A}}$, on the elements of the universe $\boldsymbol{X}$. For each $\boldsymbol{x} \in \boldsymbol{X}$, its image $\boldsymbol{A}(\boldsymbol{x}) \in [\mathbf{0},\mathbf{1}]$ is called its *degree of membership* (or its *membership grade*) in $\widetilde{\boldsymbol{A}}$.

We also say that $\tilde{A}$ is a *fuzzy subset* of X if its universe is X. There are several useful crisp sets associated with a fuzzy set $\tilde{A}$ that one finds useful. The *weak α-cuts* (also written *alpha cuts*) of $\tilde{A}$are subsets of the universe, defined by

$$^{\alpha}\tilde{A} := \{x \mid x \in X \text{ and } \alpha \leq A(x)\} \subseteq X \text{ for each } \alpha \in [0,1]$$

The term *α-cut* is used to refer to the corresponding *weakα-cut*. Analogously, the *strong α-cuts* are defined by

$$\boxed{^{\boldsymbol{\alpha}+}\widetilde{\boldsymbol{A}} := \{\boldsymbol{x} | \boldsymbol{x} \in \boldsymbol{X} \textbf{ and } \boldsymbol{\alpha} < \boldsymbol{A}(\boldsymbol{x})\} \subseteq \boldsymbol{X}, \textbf{ for each } \boldsymbol{\alpha} \in [\mathbf{0},\mathbf{1}]}$$

The *support* and *core* of $\tilde{A}$, denoted supp $\tilde{A}$ and core $\tilde{A}$ respectively, are special α-cuts. They are defined as

$$\text{supp } \tilde{A} = {}^{0+}\tilde{A} \text{and } \quad \text{core}\tilde{A} = {}^{1}\tilde{A}$$

The *heighth* of a fuzzy set is defined by

$$h(\tilde{A}) := \sup_X A$$

A *normal* fuzzy set is one whose height is equal to one.
A fuzzy set is called *convex* if and only if all its weak α-cuts are convex sets. Equivalently, one can show that the fuzzy set $\tilde{A}$ is convex iff

$$\text{for all } \ t \in [0,1] \ \text{ and } \ u, v \in X, A(tu + (1-t)v) \geq \min\{A(u), A(v)\}.$$

2.2. Zadeh's extension principle

Let $\boldsymbol{f}$ be a real valued function of $\boldsymbol{n}$ real variables, i.e., $\boldsymbol{f}$ is a map of $\mathbb{R}^{\boldsymbol{n}}$ into $\mathbb{R}$. The *fuzzy extension* of $\boldsymbol{f}$ is defined as the map that is *extended* to admit real fuzzy inputs to produce a fuzzy set as output. To be more precise, suppose that $\widetilde{\boldsymbol{A}}_{\boldsymbol{1}}, \widetilde{\boldsymbol{A}}_{\boldsymbol{2}}, \cdots, \widetilde{\boldsymbol{A}}_{\boldsymbol{n}}$ are the fuzzy inputs to $\boldsymbol{f}$. These are fuzzy subsets of $\mathbb{R}$. Write $\widetilde{\boldsymbol{B}} = \boldsymbol{f}(\widetilde{\boldsymbol{A}}_{\boldsymbol{1}}, \widetilde{\boldsymbol{A}}_{\boldsymbol{2}}, \cdots, \widetilde{\boldsymbol{A}}_{\boldsymbol{n}})$, where $\widetilde{\boldsymbol{B}}$ is also a fuzzy subset of $\mathbb{R}$ (in this paper, we shall not distinguish symbolically between a crisp function $\boldsymbol{f}$and its fuzzy extension, for brevity).
Zadeh [2, 8] defines the fuzzy extension through his *extension principle* by

$$\boldsymbol{B}(\boldsymbol{y}) = \mathbf{sup}\left\{\min_{\mathbf{1} \leq \boldsymbol{i} \leq \boldsymbol{n}} \boldsymbol{A}_{\boldsymbol{i}}(\boldsymbol{x}_{\boldsymbol{i}}) \mid \boldsymbol{f}(\boldsymbol{x}_{\mathbf{1}}, \boldsymbol{x}_{\mathbf{2}}, \cdots, \boldsymbol{x}_{\boldsymbol{n}}) = \boldsymbol{y}, \text{where each } \boldsymbol{x}_{\boldsymbol{i}} \in \mathbb{R}\right\}, \text{for all } \boldsymbol{y} \in \mathbb{R}.$$

with the understanding that $\mathbf{sup}\, \emptyset = \mathbf{0}$.

2.3. Fuzzy numbers

Fuzzy numbers [2, 15] are fuzzy subsets of the real line $\mathbb{R}$ with special properties. $\widetilde{\boldsymbol{A}}$ is called a *fuzzy number* if

(i) $\tilde{A}$ has bounded support, and is normal

(ii) All of its weak α-cuts are closed intervals in $\mathbb{R}$ for $\alpha \in (0,1]$.

Fuzzy numbers are convex fuzzy sets. This immediately follows from (ii) above, since intervals are convex.

2.4. Operations on fuzzy numbers

We use the notation that was developed earlier in Sec. 2.2. We now suppose that*f*is a continuous map and that the inputs to f are fuzzy numbers rather than fuzzy sets. In principle,Zadeh's extension principle can be used, but it is extremely difficult to use and implement. Thus, we resort to a different approach which is much easier to implement. This result is due to Buckley and Qu [12] and uses weak α-cuts. Assume that the fuzzy inputs $\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n$ to f are fuzzy numbers. Write $\tilde{B} = f(\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n)$. Then, the fuzzy set $\tilde{B}$ is a *fuzzy number*, with

$$^{\alpha}\tilde{B} = \{\, b \in \mathbb{R} \mid b = f(a_1, a_2, \cdots, a_n) \text{ with} a_i \in {}^{\alpha}\tilde{A}_i \text{for each } i \,\}, \text{for } 0 < \alpha \leq 1.$$

Annals of R.S.C.B., ISSN:1583-6258, Vol. 25, Issue 4, 2021, Pages. 5775 - 5783
Received 05 March 2021; Accepted 01 April 2021.

This result enables us to carry out a wide variety of operations on fuzzy numbers, in general. For instance, one can derive formulae for simple binary operations on fuzzy numbers.

3. Trapezoidal Fuzzy Numbers

A *trapezoidal* fuzzy number $\tilde{T}$ [15] is one whose membership function takes the form

$$T(x) = \begin{cases} \frac{x-t_1}{t_2-t_1} & \text{for } t_1 \leq x \leq t_2 \\ 1 & \text{for } t_2 \leq x \leq t_3 \\ \frac{t_4-x}{t_4-t_3} & \text{for } t_3 \leq x \leq t_4 \end{cases}, \text{ and zero otherwise}$$

for real numbers $t_i \in \mathbb{R}$, $i \in \{1,2,3,4\}$, with $t_1 < t_2 \leq t_3 < t_4$. It is customary to denote $\tilde{T} = (t_1, t_2, t_3, t_4)$ for purposes of brevity. If $t_2 = t_3$, $\tilde{T}$ is called a *triangular* fuzzy number.

It immediately follows from the definition of$T$ that supp $\tilde{T} = (t_1, t_4)$ and thatcore $\tilde{T} = [t_2, t_3]$. It is also easy to show that the α-cut of $\tilde{T}$ is given by

$${}^{\alpha}\tilde{T} = [t_1 + (t_2 - t_1)\alpha, t_4 - (t_4 - t_3)\alpha] \text{ for } \alpha \in (0,1]$$

4. (Fuzzy) Cyclic queueing networks

4.1. Basic description

The (fuzzy) queueing system that we shall deal with in this paper is a special type of a *Jackson network* (see Gross et al., [3]). Loosely speaking, a Jackson network is a collection of $N \geq 1$*nodes*, where each node represents a service facility. Each service facility can consist of multiple servers, say s_i servers at node i for $1 \leq i \leq N$. Customers can arrive at any node and depart from any node, traversing from one node to another along the way. There are no restrictions on the paths that the customers can take in the most general case. The service discipline that the queues follow is *first-come, first-served* (FCFS). Arrivals from outside the network are assumed to be *Poisson*, and service times (also called *holding times*) at nodes are assumed to be *exponentially* distributed.

The special type of a Jackson network in which no customer can enter from outside the system is referred to as a *closed* Jackson network. These systems possess a prescribed and fixed number of customers. Closed Jackson networks in which customers traverse from node to node in a sequential and circular fashion are called *cyclic queues*.

We shall restrict ourselves to cyclic queues that consist of two nodes with $s_i = 1$ for each i. Furthermore, we assume that there are C customers in the system and that service facility i functions at rate μ_i (service times are exponentially distributed as stated before) with $\mu_1 \neq \mu_2$.

The performance measures of this queueing system are already known in the literature (see Bolch et al., [13]), and we state them in the next section without proof.

4.2. Relevant Results

We denote the expected number of customers at node i in steady state by C_i^{SS}. We also write $w = \frac{\mu_1}{\mu_2} \neq 1$ for brevity. It has been established that (see [13])

$$C_1^{SS}(\mu_1, \mu_2) = \frac{C+1}{1-w^{C+1}} - \frac{1}{1-w}$$

$$C_2^{SS}(\mu_1, \mu_2) = \frac{w}{1-w} - \frac{(C+1)w^{C+1}}{1-w^{C+1}}$$

Evidently, $C_1^{SS} + C_2^{SS} = C$ as anticipated. It is important to observe that these quantities depend on the rates μ_i*only* through their ratio, w. We shall make extensive use of this fact in the coming section.

Annals of R.S.C.B., ISSN:1583-6258, Vol. 25, Issue 4, 2021, Pages. 5775 - 5783
Received 05 March 2021; Accepted 01 April 2021.

These quantities describe the performance and efficiency of the system, and hence are referred to as its *performance measures*. Our goal now is to extend these formulae to fuzzy environments, where the rates μ_i are intrinsically fuzzy in nature.

5. Solution procedure

In our analysis, we shall model the two service rates as *fuzzy numbers*. Henceforth, we shall denote these quantities by $\tilde{\mu}_1$ and $\tilde{\mu}_2$ respectively. As stated before, the performance measures of the system in question when all the variables are crisp depend only on these rates. Moreover, this dependence is only through the ratio of the rates, $\widetilde{w} := \tilde{\mu}_1/\tilde{\mu}_2$. We also demand that $1 \notin \operatorname{supp} \widetilde{w}$.

Let $\tilde{q}$ denote the (fuzzified) performance measure which is of interest to us. Our objective is to propagate fuzziness in the input fuzzy quantities, viz. the rates, to the output, viz. the performance measure of interest, using the function that relates them. Equivalently, given membership functions μ_i of the rates $\tilde{\mu}_i$, $i = 1,2$, we must construct the membership function q of $\tilde{q}$.

In principle, Zadeh's extension principle enables us to do this. We denote the function that relates the performance measure of interest and the service rates by g. Clearly, g is a real valued function of two real variables. The extension principle yields

$$q(y) = \sup\{\min\{\mu_1(x_1), \mu_2(x_2)\} \mid y = g(x_1, x_2),\ \ x_1, x_2 \in \mathbb{R}\} \text{ for all } y \in \mathbb{R}$$

with the understanding that $sup\ \emptyset = 0$. Evidently, this equation is tremendously difficult to use and implement. Therefore, we appeal to the result due to Buckley and Qu outlined in Sec. 2.4. We then immediately have

$${}^{\alpha}\tilde{q} = \{y = g(x_1, x_2) \mid x_1 \in {}^{\alpha}\tilde{\mu}_1 \text{ and } x_2 \in {}^{\alpha}\tilde{\mu}_2\} \text{ for } 0 < \alpha \leq 1$$

where $\tilde{q}$ is also a fuzzy number. It turns out that this equation serves our need very well. We can make one further simplification using the fact that $g(x_1, x_2)$ depends only on (x_1/x_2). Define the real valued map f (of a single real variable) by $f(x_1/x_2) = g(x_1, x_2)$ for all $x_1, x_2 \in \mathbb{R}$. Essentially, the expression for f, viz. $f(x)$ is obtained by replacing every occurrence of (x_1/x_2) in the expression $g(x_1, x_2)$ by x. Now, with $\widetilde{w} = \tilde{\mu}_1/\tilde{\mu}_2$, the previous equation becomes

$${}^{\alpha}\tilde{q} = \{y = f(x) \mid x \in {}^{\alpha}\widetilde{w}\} = f({}^{\alpha}\widetilde{w}) \text{ for } 0 < \alpha \leq 1$$

where $f(A)$, for $A \subseteq \mathbb{R}$, denotes the image of A under f.

Henceforth, we shall assume that $\alpha \in (0,1]$. On the other hand, since $\tilde{q}$ is a fuzzy number, ${}^{\alpha}\tilde{q}$ is a closed interval in $\mathbb{R}$. Thus, we can write ${}^{\alpha}\tilde{q} = [q_{\alpha}^{L}, q_{\alpha}^{U}]$, where

$$q_{\alpha}^{L} = \min {}^{\alpha}\tilde{q} = \min\{x \in \mathbb{R} \mid q(x) \geq \alpha\}$$

$$q_{\alpha}^{U} = \max {}^{\alpha}\tilde{q} = \max\{x \in \mathbb{R} \mid q(x) \geq \alpha\}$$

We can also write similar equations for the service rates and their ratio. We have ${}^{\alpha}\tilde{\mu}_1 = \left[\mu_{1,\alpha}^{L}, \mu_{1,\alpha}^{U}\right]$ and ${}^{\alpha}\tilde{\mu}_2 = \left[\mu_{2,\alpha}^{L}, \mu_{2,\alpha}^{U}\right]$, where $\mu_{i,\alpha}^{L} = \min {}^{\alpha}\tilde{\mu}_i$ and $\mu_{i,\alpha}^{U} = \max {}^{\alpha}\tilde{\mu}_i$ for each i, and ${}^{\alpha}\widetilde{w} = [w_{\alpha}^{L}, w_{\alpha}^{U}]$, where $w_{\alpha}^{L} = \min {}^{\alpha}\widetilde{w}$ and$w_{\alpha}^{U} = \max {}^{\alpha}\widetilde{w}$. Since the service rates are not dependent on each other, we can use standard binary interval analysis to write

$$w_{\alpha}^{L} = \frac{\mu_{1,\alpha}^{L}}{\mu_{2,\alpha}^{U}} \text{ and } w_{\alpha}^{U} = \frac{\mu_{1,\alpha}^{U}}{\mu_{2,\alpha}^{L}}$$

Putting everything together, we arrive at the following.

$$\begin{aligned} q_{\alpha}^{L} &= \min\{\, y = f(x) \mid x \in {}^{\alpha}\widetilde{\mathrm{w}}\} \\ &= \min f(x) \\ &\quad \text{subject to } x \in {}^{\alpha}\widetilde{\mathrm{w}} \end{aligned}$$

$$\begin{aligned} q_{\alpha}^{U} &= \max\{\, y = f(x) \mid x \in {}^{\alpha}\widetilde{\mathrm{w}}\} \\ &= \max f(x) \\ &\quad \text{subject to } x \in {}^{\alpha}\widetilde{\mathrm{w}} \end{aligned}$$

Equivalently, we can write

Annals of R.S.C.B., ISSN:1583-6258, Vol. 25, Issue 4, 2021, Pages. 5775 - 5783
Received 05 March 2021; Accepted 01 April 2021.

$$\begin{aligned} q_\alpha^L &= \min f(x) \\ &\quad \text{subject to } w_\alpha^L \le x \le w_\alpha^U \\ &= \min_{{}^\alpha\tilde{\mathrm{w}}} f \end{aligned}$$

$$\begin{aligned} q_\alpha^U &= \max f(x) \\ &\quad \text{subject to } w_\alpha^L \le x \le w_\alpha^U \\ &= \max_{{}^\alpha\tilde{\mathrm{w}}} f \end{aligned}$$

We have reduced the problem of determining the map q to an optimization problem. It suffices to solve the above pair of *parametric nonlinear programs*. Indeed, these are parametric nonlinear programs since the objective function f is, in general, nonlinear in its argument, and the feasible region is parametrized by a *confidence level*$\alpha \in (0,1]$. Since f is continuous on $(0,1)$ and $(1,\infty)$, and since the feasible region is a closed interval that is strictly included in exactly one of the above intervals, the extreme value theorem guarantees that the above parametric nonlinear programs are solvable in the real numbers.

We also remark that the same procedure applies to performance measures that depend on the two variables independently. The parametric nonlinear programs would then involve two decision variables rather than one.

We also define q_0^L and q_0^U as the numbers obtained by substituting zero for α in the expressions for q_α^L and q_α^U that are obtained as the solutions to the programs. These numbers are the endpoints of the support of the fuzzy number $\tilde{q}$.

It is easily seen that the collection of alpha cuts of any fuzzy set possess a *nested structure*, i.e. for any fuzzy set $\tilde{A}$ and for real numbers $0 \le \alpha < \beta \le 1$, we have that ${}^\beta\tilde{A} \subseteq {}^\alpha\tilde{A}$. For fuzzy numbers, this inclusion is necessarily strict. In particular, for the fuzzy number $\tilde{q}$, we have

$${}^\beta\tilde{q} = \left[q_\beta^L, q_\beta^U\right] \subset \left[q_\alpha^L, q_\alpha^U\right] = {}^\alpha\tilde{q}$$

Consider the maps $\alpha \mapsto q_\alpha^L$ and $\alpha \mapsto q_\alpha^U$. These are strictly increasing and strictly decreasing maps respectively, and therefore are injective. Thus, one can talk of their inverses, defined on their respective ranges. These inverses $L : [q_0^L, q_1^L] \to [0,1]$ and $R : [q_1^U, q_0^U] \to [0,1]$ constitute the membership function of the fuzzy performance measure $\tilde{q}$. Indeed, we have that

$$q(x) = \begin{cases} L(x), & q_0^L \le x \le q_1^L \\ 1, & q_1^L \le x \le q_1^U \\ R(x), & q_1^U \le x \le q_0^U \end{cases}, \text{ and zero otherwise.}$$

Analytic closed-form expressions for the functions L and R are not always possible to obtain. On the other hand, observe that the collection of intervals

$$\{[q_\alpha^L, q_\alpha^U] \mid \ \alpha \in [0,1]\}$$

can be used to obtain the graph of the map q. A finite set of these intervals can be used to arrive at an approximate plot of the membership function.

Oftentimes, one is not interested in the membership function of the fuzzy performance measure, but rather is interested in working with a crisp value that is representative of the profile of the membership function. This is called *defuzzification*, and there are a wide range of defuzzification techniques available in the literature [10]. In this paper, we use the defuzzification scheme motivated by the *Yager ranking index* [9] for ordering fuzzy numbers. The Yager ranking index of a fuzzy number $\tilde{A}$, denoted $\mu(\tilde{A})$, is defined by the equation

$$\mu(\tilde{A}) = \int_0^1 \left(\frac{A_\alpha^L + A_\alpha^U}{2}\right) \mathrm{d}\,\alpha$$

where $[A_\alpha^L, A_\alpha^U]$ is the alpha cut of $\tilde{A}$.

Finally, we defuzzify the fuzzy performance measures of the cyclic queue in discussion using the Yager ranking index, for practical use.

In the next section, we illustrate the solution procedure by means of an example.

Annals of R.S.C.B., ISSN:1583-6258, Vol. 25, Issue 4, 2021, Pages. 5775 - 5783
Received 05 March 2021; Accepted 01 April 2021.

6. Numerical Example

We shall assume that the number of customers C in the system is $C = 5$. Further, we assume that the two fuzzy rates of service, $\tilde{\mu}_1$ and $\tilde{\mu}_2$, are trapezoidal fuzzy numbers with values $\tilde{\mu}_1 = (9,10,11,12)$ and $\tilde{\mu}_2 = (3,4,5,6)$. Their corresponding alpha cuts (cf. Sec. 3) are

$$^{\alpha}\tilde{\mu}_1 = [9+\alpha, 12-\alpha], \quad ^{\alpha}\tilde{\mu}_2 = [3+\alpha, 6-\alpha]$$

for $0 < \alpha \leq 1$. Using the notation that was set up in the last section, we have

$$(\mu_1)_\alpha^L = 9+\alpha, (\mu_1)_\alpha^U = 12-\alpha \text{ and } (\mu_2)_\alpha^L = 3+\alpha, (\mu_2)_\alpha^U = 6-\alpha$$

Now, the alpha cut of the ratio of these fuzzy rates, $\widetilde{w}$, is given by

$$^{\alpha}\widetilde{w} = \left[\frac{9+\alpha}{6-\alpha}, \frac{12-\alpha}{3+\alpha}\right] \implies w_\alpha^L = \frac{9+\alpha}{6-\alpha},\ w_\alpha^U = \frac{12-\alpha}{3+\alpha}$$

The fuzzy performance measures that we shall determine are the mean number of customers in the first service station and in the second service station, $\tilde{C}_1^{SS}$ and $\tilde{C}_2^{SS}$, respectively. Denote by f_i, the crisp function that relates the ratio of the service rates x with the mean number of customers in service station i. Then, we have

$$f_1(x) = \frac{6}{1-x^6} - \frac{1}{1-x}$$
$$f_2(x) = \frac{x}{1-x} - \frac{6x^6}{1-x^6}$$

We are to construct the membership functions C_1^{SS} and C_2^{SS} of the performance measures. From the results established in the previous section, we know that the alpha cuts of the two performance measures can be obtained as solutions to a pair of parametric nonlinear programs. They are given by (henceforth, $0 < \alpha \leq 1$)

$$(C_1^{SS})_\alpha^L = \min\ f_1(x) \quad (C_1^{SS})_\alpha^U = \max\ f_1(x)$$
$$\text{subject to } w_\alpha^L \leq x \leq w_\alpha^U \qquad \text{subject to } w_\alpha^L \leq x \leq w_\alpha^U$$
$$= \min\left(\frac{6}{1-x^6} - \frac{1}{1-x}\right) \qquad = \max\left(\frac{6}{1-x^6} - \frac{1}{1-x}\right)$$
$$\text{subject to } \frac{9+\alpha}{6-\alpha} \leq x \leq \frac{12-\alpha}{3+\alpha} \qquad \text{subject to } \frac{9+\alpha}{6-\alpha} \leq x \leq \frac{12-\alpha}{3+\alpha}$$

$$(C_2^{SS})_\alpha^L = \min\ f_2(x) \quad (C_2^{SS})_\alpha^U = \max\ f_2(x)$$
$$\text{subject to } w_\alpha^L \leq x \leq w_\alpha^U \qquad \text{subject to } w_\alpha^L \leq x \leq w_\alpha^U$$
$$= \min\left(\frac{x}{1-x} - \frac{6x^6}{1-x^6}\right) \qquad = \max\left(\frac{x}{1-x} - \frac{6x^6}{1-x^6}\right)$$
$$\text{subject to } \frac{9+\alpha}{6-\alpha} \leq x \leq \frac{12-\alpha}{3+\alpha} \qquad \text{subject to } \frac{9+\alpha}{6-\alpha} \leq x \leq \frac{12-\alpha}{3+\alpha}$$

We define $\Lambda(\alpha) := \left[\frac{9+\alpha}{6-\alpha}, \frac{12-\alpha}{3+\alpha}\right] (= {}^{\alpha}\widetilde{w})$ for purposes of brevity. This is the feasible region of the parametric nonlinear programs. We also observe that $\Lambda(\alpha) \subseteq [1.5,4]$.

Essentially, these programs are global single variable optimization problems, and it turns out that tools from single variable calculus suffice to determine their solutions. The use of a computing utility like MATLAB R2020a reveals the following features of the functions f_1 and f_2, that we shall exploit.

1. The first derivative of the function f_1 takes only negative values on $[1.5,4] \supseteq \Lambda(\alpha)$. It follows that f_1 is strictly decreasing on $\Lambda(\alpha)$.

2. The first derivative of the function f_2 takes only positive values on $[1.5,4] \supseteq \Lambda(\alpha)$. It follows that f_2 is strictly increasing on $\Lambda(\alpha)$.

Thus, $f_1(x)$ attains its maximum and minimum values on $\Lambda(\alpha)$ at the points

Annals of R.S.C.B., ISSN:1583-6258, Vol. 25, Issue 4, 2021, Pages. 5775 - 5783
Received 05 March 2021; Accepted 01 April 2021.

$$x = \frac{9+\alpha}{6-\alpha} \quad \text{and} \quad x = \frac{12-\alpha}{3+\alpha}$$

respectively. Similarly, $f_2(x)$ attains its maximum and minimum values on $\Lambda(\alpha)$ at the points

$$x = \frac{12-\alpha}{3+\alpha} \quad \text{and} \quad x = \frac{9+\alpha}{6-\alpha}$$

for $0 < \alpha \leq 1$. Thus, the alpha cuts of the two performance measures are

$${}^{\alpha}\tilde{C}_1^{SS} = \left[f_1\left(\frac{12-\alpha}{3+\alpha}\right), f_1\left(\frac{9+\alpha}{6-\alpha}\right)\right] \quad \text{and} \quad {}^{\alpha}\tilde{C}_2^{SS} = \left[f_2\left(\frac{9+\alpha}{6-\alpha}\right), f_2\left(\frac{12-\alpha}{3+\alpha}\right)\right]$$

which simplify to

$${}^{\alpha}\tilde{C}_1^{SS} = \left[\frac{3+\alpha}{9-2\alpha} - \frac{6(3+\alpha)^6}{(12-\alpha)^6-(3+\alpha)^6},\ \frac{6-\alpha}{3+2\alpha} - \frac{6(6-\alpha)^6}{(9+\alpha)^6-(6-\alpha)^6}\right]$$

$${}^{\alpha}\tilde{C}_2^{SS} = \left[\frac{6(9+\alpha)^6}{(9+\alpha)^6-(6-\alpha)^6} - \frac{9+\alpha}{3+2\alpha},\ \frac{6(12-\alpha)^6}{(12-\alpha)^6-(3+\alpha)^6} - \frac{12-\alpha}{9-2\alpha}\right]$$

Now, it remains to construct the membership functions associated with the performance measures. To this end, we determine the collection of intervals

$$\{[p_\alpha^L, p_\alpha^U] \mid \ \alpha \in \{0.0, 0.1, \cdots, 1.0\}\}$$

for $p = C_1^{SS}$ and C_2^{SS},using the expressions obtained for the alpha cuts using the parametric nonlinear programs. These intervals can then be used for interpolation. Here, we have chosen eleven values for alpha that are equally spaced, and the intervals for each of these confidence levels have been calculated. The results are tabulated in Table 1.

Table 1: Results

	$[(C_1^{SS})_\alpha^L, (C_1^{SS})_\alpha^U]$	$[(C_2^{SS})_\alpha^L, (C_2^{SS})_\alpha^U]$
$\alpha = 0.0$	$[0.3319, 1.4226]$	$[3.5774, 4.6681]$
$\alpha = 0.1$	$[0.3504, 1.3623]$	$[3.6377, 4.6496]$
$\alpha = 0.2$	$[0.3697, 1.3040]$	$[3.6960, 4.6303]$
$\alpha = 0.3$	$[0.3898, 1.2475]$	$[3.7525, 4.6102]$
$\alpha = 0.4$	$[0.4108, 1.1929]$	$[3.8071, 4.5892]$
$\alpha = 0.5$	$[0.4327, 1.1402]$	$[3.8598, 4.5673]$
$\alpha = 0.6$	$[0.4556, 1.0894]$	$[3.9106, 4.5444]$
$\alpha = 0.7$	$[0.4794, 1.0405]$	$[3.9595, 4.5206]$
$\alpha = 0.8$	$[0.5043, 0.9935]$	$[4.0065, 4.4957]$
$\alpha = 0.9$	$[0.5304, 0.9482]$	$[4.0518, 4.4696]$
$\alpha = 1.0$	$[0.5575, 0.9048]$	$[4.0952, 4.4425]$

Now, one can use interpolation techniques on these intervals (in Table 1) and arrive at approximate plots of the membership functions. Here, we have used MATLAB R2020b (and linear interpolation) to arrive at the graphs (Fig. 1, 2) of the membership functions of the performance measures.

Annals of R.S.C.B., ISSN:1583-6258, Vol. 25, Issue 4, 2021, Pages. 5775 - 5783
Received 05 March 2021; Accepted 01 April 2021.

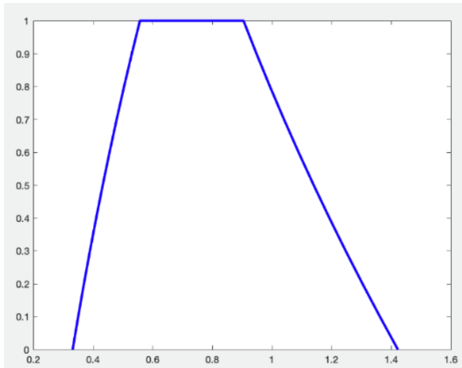

Fig. 1: Graph of the map C_1^{SS}

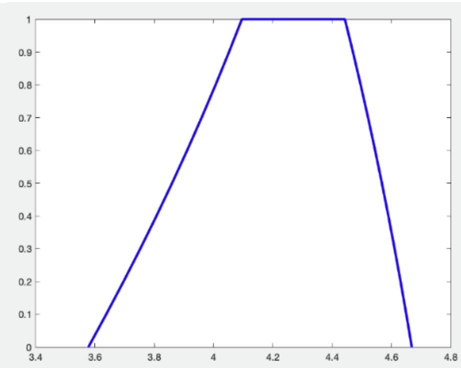

Fig. 2: Graph of the map C_2^{SS}

For practical use, these fuzzy memberships are often defuzzified into crisp values using various defuzzification techniques. In this paper, we use the defuzzification method induced by the Yager ranking index (cf. Sec. 5). Using the notation that was set up earlier, we have

$$\mu(\tilde{C}_1^{SS}) = \int_0^1 \frac{(C_1^{SS})_\alpha^L + (C_1^{SS})_\alpha^U}{2}\, \mathrm{d}\alpha \quad \text{and} \quad \mu(\tilde{C}_2^{SS}) = \int_0^1 \frac{(C_2^{SS})_\alpha^L + (C_2^{SS})_\alpha^U}{2}\, \mathrm{d}\alpha$$

Using the expressions for the alpha cuts and a computing utility (MATLAB R2020a), we arrive at the following defuzzified values for the expected number of customers in each of the two service stations.

$$\mu(\tilde{C}_1^{SS}) = 0.7924 \quad \text{and} \quad \mu(\tilde{C}_2^{SS}) = 4.2076$$

This data is extremely useful in the design of efficient queueing systems.

7. Conclusion

Cyclic queueing systems are used to model various production and service industry problems – ship operations, underground coal mining and machine repair, to name a few. Other applications are in the areas of communication networks, computer time-sharing and multiprogramming systems. This paper discusses the analysis of such systems, taking into account the fact that real world data is imprecise, and reduces the problem of determining various characteristics associated with such systems into optimization problems that are well studied in the literature, namely parametric nonlinear programs. The solutions to these programs are also shown to exist. Moreover, a defuzzification scheme is used to obtain crisp values for practical use. This analysis can easily be extended to include a wider class of queueing systems. Also, any arbitrary function of the system rates can be fuzzified using a procedure very similar to the one described in the paper. The data that is obtained through the analysis is extremely useful in the design of efficient queueing systems.

References

[1] Shih-Pin Chen, A bulk arrival queueing model with fuzzy parameters and varying batch sizes, Applied Mathematical Modelling, 30 (2006) 920-929.
[2] George, J. Klir and Bo Yuan, Fuzzy Sets and Fuzzy logic, Prentice Hall of India Private Limited (2006).
[3] Gross, D and Harris, C. M, Fundamentals of Queueing Theory, 4th ed., Wiley, New York (2008).
[4] Buckley, J. J., Feuring, T., Hayashi, Y. Fuzzy queueing theory revisited, Int. J. Uncertain. Fuzz. 9 (2001) 527-537.
[5] Negi, D. S., Lee, E. S. Analysis and simulation of fuzzy queues, Fuzzy Sets Syst. 46 (1992) 321-330.
[6] Li, R. J., Lee, E. S. Analysis of fuzzy queues, Comput. Math. Appl. 17 (1989) 1143-1147.
[7] Zadeh, L. A. Fuzzy sets, Inform. Contr. (1965) 338-353.
[8] Zadeh, L. A. The concept of linguistic variable and its applications to approximate reasoning, Inform. Sci. 8 (1975) 199-249, 301-357; 9 (1975) 43-80.
[9] Ronald, R., Yager, A procedure for ordering fuzzy subsets of the unit interval, Information Sciences, vol. 24(2), (1981) 143-161.
[10] Philippe Fortemps, Marc Roubens, Ranking and defuzzification methods based on area compensation, Fuzzy Sets and Systems, vol. 82(3), (1996) 319-330.
[11] Kao, C., Li, C. C. and Chen, S. P. Parametric Programming to the analysis of queues, Fuzzy sets and systems, vol. 107, (1999) 93-100.
[12] Buckley, J. J., Yunxia Qu. On using α-cuts to evaluate fuzzy equations, Fuzzy Sets and Systems, vol. 38(3), (1990) 309- 312.
[13] Gunter Bolch, Stefan Greiner et al., Queueing networks and Markov Chains, 2nd edition, A. John Wiley and Sons, Inc., Publications (2006).

Annals of R.S.C.B., ISSN:1583-6258, Vol. 25, Issue 4, 2021, Pages. 5775 - 5783
Received 05 March 2021; Accepted 01 April 2021.

[14] Dong, W. M., Shah, H. C., Wongt, F. S. Fuzzy computations in risk and decision analysis, Civil Engineering Systems, vol. 2(4) (1985) 201-208.
[15] Hanss, M. Applied Fuzzy Arithmetic. An Introduction with Engineering Applications. Springer- Verlag, Berlin Heildelberg (2005).
[16] Dubois, D., Prade, H. Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York (1980).
[17] Mizumoto, M., and Tanaka, K. Some properties of fuzzy numbers, Advances in Fuzzy Set Theory and Applications. NLD; Amsterdam: North Holland; DA, (1979) 153-164; Bibl. 3 ref.
[18] MukebaKanyina, J. P. et al., Computing Fuzzy Queueing Performance Measures by L-R Method, Journal of Fuzzy Set Valued Analysis, 1 (2015) 57-67.
[19] Nakumara, Y. Extension of algebraic calculus on fuzzy numbers using α level sets, FuzzyInformation Processing (FIP-84) Symposium, Kauai, Hawaii (1984).
[20] Kaufmann, A. Introduction to the Theory of Fuzzy Subsets. Vol. 1: Fundamental TheoreticalElements, Academics Press, New York (1975).