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# Transient Analysis of Single Server Queueing System with Glue Periods

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## ABSTRACT

In this work, we investigate a single server queueing system with glue periods. The arrival stream is governed by a Poisson process, while service times and glue periods are assumed to be exponentially distributed. The glue period is triggered just before the arrival of the customer at the station. During the glue period, customers arriving at the station stick to the station's queue and will be served during the service period of the station. The glue period is when customers can also make a reservation at a station for service in the subsequent service period of that station. We derive the probability distribution at random points and at departure points and other performance indices such as the average number of customers and the average waiting time in the queue and the system by applying the Laplace Transform technique. The time-dependent performance measures of the system are examined. The related steady-state investigation and key performance measures of the system are likewise exhibited. Finally, we validate our analytical results by some numerical examples and study the impact of parameters on the system's performance characteristics.

**Keywords**
Queueing System, Networks, Transient analysis, Waiting Time, Busy Period, Glue Period, Laplace transforms.

## 1. Introduction

In recent years, there has been an increased interest in queuing systems and their applications. This is mainly because queuing models naturally arise in the performance analysis of a wide range of systems in data distributed networks, telecommunications, and traffic management on high-speed networks and production engineering [1, 15, 24]. New technological advances in computer systems and data communication networks have often inspired new results in queuing systems. The methods of queueing networks have always been a fundamental component of the study of communication systems. The widespread introduction of computers into these systems has introduced new results on queueing networks in studies of large communication networks' performance. Some of the other prominent applications of the queueing theory are landing aircraft, loading, unloading of ships, machine repair, manufacturing process, taxi services, supply chain management and toll booths.

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Several analytical results have been discussed for many queuing models under steady-state conditions. But analytical results for queuing systems' transient behavior are not as widely available as steady-state results. The latter are well suited to discuss the system's performance on a long time scale, while the former is useful for investigating the dynamical behavior of the systems over a finite time horizon. Further, transient analysis helps us understand the behavior of a system when the parameters involved are perturbed and can contribute to the costs and benefits of operating systems. Moreover, such transient analysis is instrumental in obtaining optimal solutions that lead to the system's control. Despite its broad interest, the analytical approach to describing queuing systems' transient behavior is notably difficult because of their mathematical intractability. Although much effort has been devoted to determining the exact time-dependent queue behavior, very few useful general results exist. Thus, the investigation of the transient analysis of queuing processes is highly essential from the point of view of the theory and its applications.

This motivated us to investigate the transient analysis of a single server queuing system with glue period. This paper is organized in the following manner: In the next section 2, the mathematical model is described and explicit expressions for the transient probabilities of the system are discussed. The mean number and workload of the system are given in Section 3. Section 4 discusses the steady-state probabilities of the system under investigation. Some key performance measures under the steady-state condition are listed out in Section 5. A numerical example is presented to illustrate the effect of system parameters on the performance measures in Section 6. Finally, Section 7 concludes the article.

### 1.1 Literature survey

Transient analysis is very useful in obtaining optimal solutions which lead to the control of the system. Krishna Kumar and Arivudainambi [17], Krishna Kumar et al. [14, 16] have discussed an $M \mid M \mid 1$ queuing system with catastrophes. They have obtained explicit expressions for the transient and steady-state probabilities of the system size and related performance measures. Recent years have brought a rapidly increasing demand for real-time services in which jobs have specific timing requirements. Examples of such services include voice and video transmission, manufacturing systems, where the orders have due dates, real-time control systems and tracking systems. Another important class of applications arises in medical scheduling problems, like organ allocation or prioritizing admissions to emergency rooms.

Multi-server preemptive priority systems with two customer classes have also received some attention in the literature. The multiprocessor system has wide applications in telecommunications, flexible manufacturing systems, reliability and traffic control. Many researches on the analysis of queueing system with multi-server, primarily homogeneous servers, are available in order to deal with the the behavior of multiprocessor systems. Although the difficulty of obtaining time-dependent solutions to queueing systems is well known, many researchers have studied such cases; see [2] and references therein. For related literature, interested readers may refer to [20] and references therein. Trivedi [23] analyzed the multiprocessor system consisting of two types of processors where one processor is faster than the other and no queue is allowed in front of the slower processor. He restricts his analysis to steady-state measures. For the same model, Dharmaraja [7] obtained the exact time dependent system size probabilities.

Mostly Researchers study both transient and steady-state distributions of queueing systems in continuous time. This is mainly due to the fact that queueing models are used for the performance and reliability of a wide range of systems in wireless networks, telecommunication systems and data distributed networks. Now a days, new results in queueing systems have often been inspired by latest technical advances in computer network systems and wireless communication systems. For design and tuning of the advanced network system, performance must be analysed mathematically and evaluated numerically. For excellent overview of the fundamental techniques and classical results on queueing theory, we refer the reader to the monographs by Bertsekas and Gallager [5], Takagi [22], Daigle [6] Gelenbe and Pujolle [9], Hayes and GaneshBabu [10], Giambene [3], Yadavalli and Anbazhagan [27] and Anantha Lakshmi et al. [18]. The analyses in these papers mostly yield implicit expressions for performance characteristics through Laplace transforms, integro-differential equations and infinite convolutions. More specifically, there is vast literature on the transient analysis of the $M|M|1$ queue with the goal to derive explicit expressions for queue length characteristics.

Recently, Parthasarathy and Sudesh [21] have studied an $M|M|C$ queuing system with the widely discussed N-policy [28]. The transient solution for the system size probabilities and busy period distribution of the system are derived for the model under discussion. Wireless networks have received more attention as a mean of data communication among portable devices. As wireless devices usually rely on portable power sources such as batteries to provide the necessary operational power, power management in wireless networks has become a critical issue. For instance, IEEE 802.16e is designed to support high capacity, high data rate and multimedia services as an emerging broadband wireless access system for fixed and mobile service stations. Several researchers have been developed analytical models and obtained the performance of the sleep mode operations in the IEEE 802.16e system (see Hwang, Kim, Son, and Choi [12, 13] 2009, 2010; Baek, Son,and Choi [4] v 2011; Huo, Jin, and Wang [11] 2011). These wireless network characteristics can be modeled from a more practical viewpoint if the transient analysis can be integrated.

## 2. Model Description

We consider an $M|M|1$ queueing system with unlimited capacity waiting room for customers to wait. Let the arrival of customers follow a Poisson process with rate $\lambda$ and service times follow an exponential distribution with

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rate $\mu$. It is assumed that the glue period is exponentially distributed with rate $\theta$. The glue period is activated just before the arrival of the server at the station. During glue period, customer arriving at the station stick to the queue of the station and will be served during the service period of that station. It also include the period period in which customers can make a reservation at a station for service in the subsequent service period of that station. This can also be done when there are no customers in the system. The glue period can also be considered as repair time or a vacation queueing model with single vacation. We model our single server queueing with glue period as a continuous time Markov chain. Let $\{X(t); t\grave{o}R^+\}$ be the continuous time chain random process, $P_n(t) = P(X(t) = n)$, $n = 0,1,2,...$ denotes the probabilities that there are $n$ customers in the system at time t when the server is in working state/on state and $G(t) = P(X(t) = G)$ is the probability that the server is in glue period at time $t$. It is clear that the state space of the system is $S = G,0,1,2,...$. Let

$$P(z,t) = G(t) + \sum_{n=0}^{\infty} P_n(t) z^n$$

be the probability generating function and $m(t) = E\left[X(t)\right]$ be the mean number of customers at time $t$ for the model under discussion.

From the above assumption it is clear that the system size probabilities $P_n(t)$, n=0,1,2,.. the glue period state probability $G(t)$ of the server satisfy the following Chapman-Kolmogorov equation:

$$\frac{dG(t)}{dt} = -(\theta + \lambda)G(t) + \mu P_1(t) \tag{1}$$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \theta G(t) \tag{2}$$

$$\frac{dP_1(t)}{dt} = -(\lambda + \mu)P_1(t) + \lambda P_0(t) + \mu P_2(t) + \lambda G(t) \tag{3}$$

$$\frac{dP_n(t)}{dt} = -(\lambda + \mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t), \quad n = 2,3,4.. \tag{4}$$

$$P_0(0) = 1, P_n(0) = 0, n \neq 0, \ C(0) = 0 \quad and \quad V(0) = 0 \tag{5}$$

Multiplying equations (3) and (4) by $z$ and $z^n$ respectively summing over all the values of $n$, and then using

$$P(z,t) = G(t) + \sum_{n=0}^{\infty} P_n(t) z^n$$

it is seen that the probability generating function $P(z,t)$ satisfies the differential equation

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$$\frac{dP(z,t)}{dt} = \left(\lambda z - (\lambda+\mu) + \frac{\mu}{z}\right)P(z,t) + \mu\left(1-\frac{1}{z}\right)\left[P_0(t)+G(t)\right] \tag{6}$$

with the initial condition $P(z,0)=1$ (7)

the solution of (6) can be obtained as

$$P(z,t) = e^{[\lambda z-(\lambda+\mu)+\frac{\mu}{z}]t} + \int_0^t \left[\mu(1-\frac{1}{z})P_0(u)+G(u)\right]e^{[\lambda z-(\lambda+\mu)+\frac{\mu}{z}](t-u)du} \tag{8}$$

It is well known that if $\alpha = 2\sqrt{\lambda\mu}$ and $\beta = \sqrt{\frac{\lambda}{\mu}}$, then

$$exp\left(\lambda z + \frac{\mu}{z}\right)t = \sum_{n=-\infty}^{\infty} I_n(\alpha t)(\beta z) \tag{9}$$

where $I_n(\alpha t)(\beta z)$ is the modified Bessel function of order $n$ (see [25]). Using (9) in equation (8) and comparing the coefficient of $z^n$ on either side we get for $n=1,2,3,...$

$$\begin{aligned} P_n(t) = \{&\beta^n e^{-(\lambda+\mu)t} I_n(\alpha t) + \mu\beta^n \int_0^t [P_0(u)+G(u)]e^{-(\lambda+\mu)(t-u)} I_n(\alpha(t-u))du \\ &- \mu\beta^{n+1}\int_0^t [P_0(u)+G(u)]e^{-(\lambda+\mu)(t-u)}du\} \end{aligned} \tag{10}$$

As $P(z,t)$ does not contain terms with negative powers of $z$, the right hand side of equation (10) with $n$ replaced by $-n$ must be zero. Thus,

$$\begin{aligned} 0 = &\beta^{-n}e^{-(\lambda+\mu)t}I_{-n}(\alpha t) + \mu\beta^{-n}\int_0^t \left[P_0(u)+G(u)\right]e^{-(\lambda+\mu)(t-u)}I_{-n}(\alpha(t-u))du \\ &- \mu\beta^{-n+1}\int_0^t \left[P_0(u)+G(u)\right]e^{-(\lambda+\mu)(t-u)}I_{-n+1}(\alpha(t-u))du \end{aligned}$$

Multiplying the above equation by $\beta^{2n}$ on both sides and after some algebra we get

$$\begin{aligned} &\beta^n e^{-(\lambda+\mu)t}I_n(\alpha t) + \int_0^t [P_0(u)+G(u)]e^{-(\lambda+\mu)(t-u)}I_n[\alpha(t-u)]du \\ &= \mu\beta^{n+1}\int_0^t \left[P_0 + G(u)\right]e^{-(\lambda+\mu)(t-u)}I_{n-1}(\alpha(t-u))du \end{aligned} \tag{11}$$

where we have used $I_{-n}(.) = I_n(.)$. On using equation (11) in equation (10) and simplyfying we get an elegant expression for $P_n(t), n=1,2,3...$ as,

$$P_n(t) = n\beta^n \int_0^t [P_0(u)+G(u)]e^{-(\lambda+\mu)(t-u)}\frac{I_n\alpha(t-u)}{(t-u)}du \tag{12}$$

We use Laplace transform with respect to time to find $G(t)$ and $P_0(t)$. For any function $f(.)$ and $f^*(s)$ denotes its Laplace transform and denote the convolution. By transforming equations (1), (2) and (12), we get

$$G^*(s) = \frac{\mu}{s+\lambda+\theta}P_1^*(s) \tag{13}$$

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$$P_0^*(s) = \frac{1}{s+\lambda} + \frac{\theta}{s+\lambda} G^*(s) \tag{14}$$

$$P_1^*(s) = \frac{(s+\lambda+\mu) - \sqrt{(s+\lambda+\mu)^2 - 4\lambda\mu}}{2\mu}[P_0^*(s) + G^*(s)] \tag{15}$$

These equations after some algebric manupulation and rearrangement yield,

$$G^*(s) = \frac{\frac{\lambda}{s+\lambda}[\frac{s+\lambda+\mu-\sqrt{(s+\lambda+\mu)^2-\alpha^2}}{2\lambda}]\frac{1}{s+\lambda+\mu}}{1-H^*(s)} \tag{16}$$

$$P_0^*(s) = \frac{1}{s+\lambda} \frac{\frac{\lambda}{s+\lambda}[\frac{s+\lambda+\mu-\sqrt{(s+\lambda+\mu)^2-\alpha^2}}{2\lambda}]\frac{\theta}{s+\lambda+\theta}\frac{1}{s+\lambda}}{1-H^*(s)} \tag{17}$$

where

$$H^*(s) = \frac{\lambda}{s+\lambda+\theta}[\frac{(s+\lambda+\mu)-\sqrt{(s+\lambda+\mu)^2-\alpha^2}}{2\lambda}] + \frac{\theta}{s+\lambda+\theta}\frac{\lambda}{s+\lambda}[\frac{(s+\lambda+\mu)-\sqrt{(s+\lambda+\mu)^2-\alpha^2}}{2\lambda}] \tag{18}$$

Equation (16) and (17) can be expressed as

$$G^*(s) = \frac{\lambda}{s+\lambda}[\frac{(s+\lambda+\mu)-\sqrt{(s+\lambda+\mu)^2-\alpha^2}}{2\lambda}] \times \sum_{n=0}^{\infty}(H^*(s))^n \frac{1}{s+\theta+\lambda} \tag{19}$$

$$P_0^*(s) = \frac{1}{s+\lambda} + \frac{\lambda}{s+\lambda}\left[\frac{(s+\lambda+\mu)-\sqrt{(s+\lambda+\mu)^2-\alpha^2}}{2\lambda}\right] \times \sum_{n=0}^{\infty}(H^*(s))^n \frac{\theta}{s+\lambda+\theta}\frac{1}{s+\lambda} \tag{20}$$

Where $H^*(s)$ is given in equation (18), Inversion of (19) and (20) yields,

$$G(t) = \int_0^t e^{-\lambda(t-u)}\lambda\beta^{-1}e^{-(\lambda+\mu)(t-u)}\frac{I_1\alpha(t-u)}{(t-u)} \times \sum_{n=0}^{\infty} H^{(n)}(u)e^{-(\lambda+\theta)u}du \tag{21}$$

$$P_0(t) = e^{-\lambda t} + \int_0^t e^{-\lambda(t-u)}\lambda\beta^{-1}e^{-(\lambda+\mu)(t-u)}\frac{I_1\alpha(t-u)}{(t-u)} \times \sum_{n=0}^{\infty} H^{(n)}(u)e^{-(\lambda+\theta)u}\theta e^{-\lambda u}du \tag{22}$$

Where $H^{(n)}(t)$ is the $n$-fold convolution of

$$H(t) = \int_0^t e^{-(\theta+\lambda)u}\lambda\beta^{-1}e^{-(\lambda+\mu)(t-u)}\frac{I_1\alpha(t-u)}{(t-u)}du + \int_0^t e^{-(\lambda+\theta)u}\theta e^{-\lambda\mu}\lambda\beta^{-1}e^{-(\lambda+\mu)(t-u)}\frac{I_1\alpha(t-u)}{(t-u)}du \tag{23}$$

with itself and

$$H^{(0)}(t) = \begin{cases} 1 & if \;\; t > 0 \\ 0 & if \;\; t \le 0 \end{cases}$$

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we summarise the main results in the following theorem.

**Theorem 1:**

The transient probabilities of system size $P_n(t), n = 0,1,2,...$ and the glue period probability $G(t)$ for the model under discussion are obtained as

$$P_0(t) = e^{-\lambda t} + \int_0^t e^{-\lambda(t-u)} \lambda \beta^{-1} e^{-(\lambda+\mu)(t-u)} \frac{I_1 \alpha(t-u)}{(t-u)} \times \sum_{n=0}^{\infty} H^{(n)}(u) e^{-(\lambda+\theta)u} \theta e^{-\lambda u} du$$

$$G(t) = \int_0^t e^{-\lambda(t-u)} \lambda \beta^{-1} e^{-(\lambda+\mu)(t-u)} \frac{I_1 \alpha(t-u)}{(t-u)} \times \sum_{n=0}^{\infty} H^{(n)}(u) e^{-(\lambda+\theta)u} du$$

$$P_n(t) = n\beta^n \int_0^t [P_0(u) + G(u)] e^{-(\lambda+\mu)(t-u)} \times \frac{I_n \alpha(t-u)}{(t-u)} du, n = 1,2,3,..$$

where $H(t)$ is given by the equation (23) and $H^{(0)}(t)$ is the $n$-fold convolution of $H(t)$ with itself

**Remark:**

From equation (12) we can express the system size probability $P_n(t), n = 1,2,3,...$ as

$$P_n(t) = \int_0^t [P_0(u) + G(u)] \left(\frac{\lambda}{\mu}\right)^n f_{n0}(t-u) du \tag{24}$$

where

$$f_{n0}(t) = n\beta^{-n} e^{-(\lambda+\mu)t} \frac{I_n(\alpha t)}{t}$$

is the probability density function of busy period of an $M \mid M \mid 1$ queue beginning at first with $n$ customers. Thus the equation (24) establish an interesting relation between the system size probability $P_n(t), n = 1,2,3,...$ and the busy period distribution of the $M \mid M \mid 1$ queue for our general queueing system.

**Theorem 2:**

The asymptotic behaviour of the probability $P_0(t)$ is as follows

1. If $\lambda < \mu,\ \theta > 0,$ then $P_0(t) \to \dfrac{\frac{\theta}{\mu}}{\frac{\theta}{\lambda} + \frac{(1+\frac{\theta}{\lambda})}{(1-\rho)}}, \quad as \quad t \to \infty \quad where \quad \rho = \dfrac{\lambda}{\mu}$ (25)

2. If $\lambda = \mu, \theta > 0,$ then $P_0(t): \dfrac{\theta}{\lambda+\theta} \dfrac{1}{\sqrt{\pi \lambda t}} \quad as \quad t \to \infty$ (26)

3. If $\lambda > \mu$ and $\theta > 0,$ then $\displaystyle\int_0^{\infty} P_0(t) dt = \frac{\theta + \lambda - \mu}{(\lambda+\theta)(\lambda-\mu)}$ (27)

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**Proof:**

From equation (17) for $\lambda \neq \mu$ and $\theta > 0$ we get

$$P_0^*(s) = \frac{\frac{1}{s+\lambda}\left(1-\frac{1}{2(s+\lambda+\theta)}\left[\left[(s+\lambda+\mu)-(\lambda-\mu)\left(1+\frac{2s(\lambda+\mu)}{(\lambda-\mu)^2}+\frac{s^2}{(\lambda-\mu)^2}\right)\right]^{\frac{1}{2}}\right)\right)}{K(s)} \quad (28)$$

where

$$K(s) = 1 - \frac{1}{2(s+\lambda+\theta)}\left[1+\frac{\theta}{s+\lambda}\right]\left[s+\lambda+\mu-(\lambda-\mu)\left(1+\frac{2s(\lambda+\mu)}{(\lambda-\mu)^2}-\frac{s^2}{(\lambda-\mu)^2}\right)^{\frac{1}{2}}\right]$$

Expanding binomially both the numerator and denominator of the above expression in powers of $s$ and taking limit as $s \to 0$, we have

$$P_0^*(s): \begin{cases} \dfrac{(\lambda+\theta-\mu)+o(s)}{\theta(\lambda-\mu)+\lambda(\lambda-\mu)+o(s)}, & \text{if} \quad \lambda > \mu \\ \dfrac{\theta+o(s)}{s[\lambda+\theta+\lambda\theta+\frac{\lambda\theta}{\lambda-\mu}\frac{\lambda^2}{\lambda-\mu}]+o(s^2)}, & \text{if} \quad \lambda < \mu \end{cases} \quad (29)$$

By using the Tauberian theorem [26] the results (25) and (27) follows from equation (29), the result for $P_0(t)$ in the case $\lambda = \mu, \theta > 0$ is obtained from equation (17) we have

$$P_0^*(s) = \frac{\frac{1}{s+\lambda}\left(1-\frac{1}{s+\lambda+\theta}\left[\frac{s+2\lambda-(s^2+4\lambda s)^{\frac{1}{2}}}{2}\right]\right)}{1-\frac{1}{2(s+\lambda+\theta)}\left[1+\frac{\theta}{s+\lambda}\right]\left[\frac{s+2\lambda-(s+4\lambda s)^{\frac{1}{2}}}{2}\right]} \quad (30)$$

Again expanding the numerator and denominator of the above expression in powers of $s$ and taking the limit as $s \to 0$ we get

$$P_0^*(s): \frac{\theta+\sqrt{\lambda}\sqrt{s}+o(s)}{\sqrt{s}\sqrt{\lambda}(\lambda+\theta)+o(s)} \quad (31)$$

Invoking the Tauberian theorem again, the result (26) follows from equation (31)

**Theorem 3:**

The asymptotic behaviour of the glue period probability $G(t)$ is as follows:

1. If $\lambda < \mu, \theta > 0$, then $G(t) = \dfrac{1}{\frac{\theta}{\lambda}+\frac{(1+\frac{\theta}{\lambda})}{(1-\rho)}}, \quad as \quad t \to \infty \quad where \quad \rho = \dfrac{\lambda}{\mu}$ (32)

2. If $\lambda=\mu, \theta>0$, then $G(t): \dfrac{1}{\theta+\lambda}\sqrt{\dfrac{\lambda}{\pi t}} \quad as \quad t\to\infty$ (33)

3. If $\lambda>\mu$ and $\theta>0$, then $\int_0^\infty G(t)dt = \dfrac{\mu}{(\lambda+\theta)(\lambda-\mu)}$ (34)

**Proof:**

For $\lambda\neq\mu$ and $\theta>0$ by rearranging the terms in equation (16), we have

$$G^*(s)=\frac{\frac{\lambda}{s+\lambda}\left(\frac{1}{(s+\lambda+\theta)}\left[(s+\lambda+\mu)-(\lambda-\mu)\left(1+\frac{2s(\lambda+\mu)}{(\lambda-\mu)^2}+\frac{s^2}{(\lambda-\mu)^2}\right)\right]^{\frac{1}{2}}\right)}{F(s)} \tag{35}$$

where

$$F(s)=1-\frac{1}{2(s+\lambda+\theta)}\left[1+\frac{\theta}{s+\lambda}\right]\left[s+\lambda+\mu-(\lambda-\mu)\left(1+\frac{2s(\lambda+\mu)}{(\lambda-\mu)^2}-\frac{s^2}{(\lambda-\mu)^2}\right)^{\frac{1}{2}}\right]$$

Expanding binomially the numerator and denominator of equation (35) in powers of we get for $\lambda<\mu$ and $\theta>0$, after some mathematical manupulation,

$$C^*(s):\ \frac{\lambda+o(s)}{s[\lambda\theta+\frac{\mu(\theta+\lambda)}{(\mu-\lambda)}]+o(s^2)}, \quad as \quad s\to 0 \tag{36}$$

and for $\lambda>\mu$ and $\theta>0$ we obtain again from equation (35)

$$C^*(s):\ \frac{\mu+o(s)}{(\lambda+\theta)(\lambda-\mu)+o(s)}, \quad as \quad s\to 0 \tag{37}$$

Similarly for $\lambda=\mu$, and $\theta>0$ expanding both the numerator and denominator in powers of $s$ equation (35) leads to

$$C^*(s):\ \frac{\lambda[1-\frac{\sqrt{s}}{\sqrt{\lambda}}+o(s)]}{\sqrt{s}(\lambda+\theta)\sqrt{\lambda}+o(s)}, \quad as \quad s\to 0 \tag{38}$$

Thus the results (32), (33) and (34) follows from equations (36), (37) and (38) by invoking Tauberian theorem [26].

## 3. Mean number and workload

The main objective of this section is to determine the time dependent performance measures such as the complementary cumulative distribution function of $X(t)$, the mean number $m(t)$ of customers present in the system and the expected work $E(W(t))$ at time $t$, where $W(t)$ is the workload or virtual waiting time at time $t$ for the queueing system understudy. By definition the complementary cumulative distribution function of the system size $X(t)$ is given by,

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$$P(X(t)\geq n)=\sum_{k=n}^{\infty}P_k(t)=\int_0^t[P_0(u)+G(u)]\sum_{k=n}^{\infty}k\beta^k e^{-(\lambda+\mu)(t-u)}\frac{I_k\alpha(t-u)}{(t-u)}du \tag{39}$$

The above equation can be written as

$$P(X(t)\geq n)=\int_0^t[P_0(u)+G(u)]\sum_{k=0}^{\infty}k\left(\frac{\lambda}{\mu}\right)^k f_{k0}(t-u)du \tag{40}$$

where $f_{k0}(t)$ is defined in equation (24) .

The mean $m(t)=E[X(t)]$ is given as

$$m(t)=\sum_{n=1}^{\infty}P(X(t)\geq n) \tag{41}$$

$$=\int_0^t[P_0(u)+G(u)]\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\left(\frac{\lambda}{\mu}\right)^k f_{k0}(t-u)du=\int_0^t[P_0(u)+G(u)]\sum_{k=1}^{\infty}k\left(\frac{\lambda}{\mu}\right)^k f_{k0}(t-u)du$$

The expected workload or virtual waiting time $E[W(t)]$ in the system at time $t$ is

$$E[W(t)]=\frac{E[X(t)]}{\mu}=\frac{1}{\mu}\int_0^t[P_0(u)+G(u)]\sum_{k=1}^{\infty}k\left(\frac{\lambda}{\mu}\right)^k f_{k0}(t-u)du \tag{42}$$

## 4. Steady-State Analysis

We will examine the behavior of the steady-state probabilities of the system size and the glue period probability of the server of our queueing system. From the perspective of practical applications, such steady-state probabilities(i.e persistent) rather than start up (i.e transient) behavior is useful.We will examine the behavior of the steady-state probabilities of the system size and the glue period probability of the server of our queueing system. From the perspective of practical applications, such steady-state probabilities(i.e persistent) rather than start up (i.e transient) behavior is useful.

**Theorem 4:**

For $\lambda<\mu$ and $\theta>0,$ the steady-state probabilities of the system size $\{P_n:n=0,1,2...\}$ the glue period probability $G$ of the queueing system are obtained as

$$G=\frac{1}{\theta+\dfrac{\lambda+\theta}{\lambda-\mu}}=\frac{1}{\theta+\dfrac{1+\dfrac{\theta}{\lambda}}{1-\rho}} \tag{43}$$

$$P_0=\frac{\dfrac{\theta}{\lambda}}{\theta+\dfrac{1+\dfrac{\theta}{\lambda}}{1-\rho}} \tag{44}$$

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and
$$P_n = \frac{(1+\frac{\theta}{\lambda})\rho^n}{\theta + \frac{(1+\frac{\theta}{\lambda})}{(1-\rho)}} \tag{45}$$

where $\rho = \frac{\lambda}{\mu}$

**Proof:**

For $\lambda < \mu, \theta > 0$ we get from equations (12) - (15)

$$P_n^*(s) = \frac{[(s+\lambda+\mu) - \sqrt{(s+\lambda+\mu)^2 - \alpha^2}]^n [P_0^*(s) + G^*(s)]}{(2\mu)^n}, \quad n = 1,2,3...$$

$$G^*(s) = \frac{\mu}{s+\lambda+\theta} P_1^*(s)$$

$$P_0^*(s) = \frac{1}{s+\lambda} + \frac{\theta}{s+\lambda} G^*(s)$$

Now using the Tauberian theorem [26] we get

$$P_n = \lim_{s\to 0} sP_n^*(s) = \frac{[(\lambda+\mu) - (\mu-\lambda)]^n (P_0 + G)}{(2\mu)^n} \tag{46}$$

$$= \left(\frac{\lambda}{\mu}\right)^n (P_0 + G), \quad n = 1,2,3,...$$

$$G = \lim_{s\to 0} sG^*(s) = \frac{\mu}{\lambda+\theta} P_1 \tag{47}$$

and

$$P_0 = \lim_{s\to 0} sP_0^*(s) = \frac{\theta}{\lambda} \tag{48}$$

After some mathematical manipulations, we get from (46) - (48)

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 + \frac{\theta}{\lambda}\right) G, \quad n = 1,2,3,... \tag{49}$$

Now using the normalising condition $G + P_0 + \sum_{n=1}^{\infty} P_n = 1$ to get the uniform probability $G$ as

$$G = \frac{1}{\theta + \frac{1+\frac{\theta}{\lambda}}{1-\rho}} \tag{50}$$

Hence equations (46) to (50) completely determine all the steady-state probabilities

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$\{P_n, n = 0,1,2,...\}$ of the system size and the steady-state probability $G$ of the glue period probability.

## 5. System Performance Measures

We are currently in a situation to study some fascinating and significant performance measures, namely, mean of the system size, accessibility of the server, system throughput, mean waiting time, mean cycle time of the system and expected number of customers served during the busy period under steady-state condition.

Based on the Theorem 4, we can get the probability generating function and the corresponding moments through the steady-state. It is seen that the steady-state moments are regularly great approximations to the transient counterpart expressions even when time t is of moderate size.

We have the following theorem.

**Theorem 5:**

If $\lambda < \mu$ and $\theta > 0$ then the steady-state probability generating function $P(z)$ of the number of customers in the system is given by

$$P(z) = E[z^X] = \frac{(1-\rho)}{(1-\rho z)}\left[\frac{1+\frac{\theta}{\lambda}+\theta(1-\rho z)}{1+\frac{\theta}{\lambda}+\theta(1-\rho)}\right] \tag{51}$$

The mean $E(X)$ of the system size is obtained as

$$E(X) = \frac{\frac{1+\frac{\theta}{\lambda}}{1-\rho}}{\theta+\frac{1+\frac{\theta}{\lambda}}{1-\rho}}\frac{\rho}{1-\rho}, \quad where \quad \rho = \frac{\lambda}{\mu} \tag{52}$$

Let $Q$ denote the number of customers in the queue. Then, the steady-state probability generating function $\psi(z)$ and the mean $E(Q)$ are determined as

$$\psi(z) = E(z^Q) = \frac{1+\frac{\theta}{\lambda}}{\theta+\frac{1+\frac{\theta}{\lambda}}{1-\rho}} + \frac{1+\frac{\theta}{\lambda}}{\theta+\frac{1+\frac{\theta}{\lambda}}{1-\rho}}\frac{\rho}{1-\rho z} \tag{53}$$

$$E(Q) = \frac{\frac{1+\frac{\theta}{\lambda}}{1-\rho}}{\frac{\theta}{\lambda}+\frac{1+\frac{\theta}{\lambda}}{1-\rho}}\frac{\rho^2}{1-\rho} \tag{54}$$

Multiplying (54) by $z^n$ and summing over $n, n = 1,2,3,...$ and then adding equations (43) and (44), after a little algebra we get (51). The result (52) follows directly from (51) on differentiation with respect to $z$ and setting $z = 1$ .

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**Corollary 1:**

The second moment $E(X^2)$ and variance $V(X)$ of the system size, under steady-state are given as

$$E(X^2) = \frac{\frac{1+\frac{\theta}{\lambda}}{1-\rho}}{\theta + \frac{1+\frac{\theta}{\lambda}}{1-\rho}} \frac{\rho(1+\rho)}{(1-\rho)^2} \tag{55}$$

and

$$Var(X) = \frac{\frac{1+\frac{\theta}{\lambda}}{1-\rho}}{\theta + \frac{1+\frac{\theta}{\lambda}}{1-\rho}} \frac{\rho(1+\rho)}{(1-\rho)^2} - \left[ \frac{\frac{1+\frac{\theta}{\lambda}}{1-\rho}}{\theta + \frac{1+\frac{\theta}{\lambda}}{1-\rho}} \frac{\rho}{(1-\rho)} \right]^2 \tag{56}$$

**Remark 2:**

It is interesting to note that stochastic decomposition law [8, 19] can be demonstrated for our queueing system also. we can write equation (51) as

$$P(z) = \prod\nolimits_{M|M|1}(z)\chi(z)$$

where $\prod_{M|M|1}(z) = \frac{1-\rho}{1-\rho z}$, the probabilty generating function of the number of customers in the $M \mid M \mid 1$ queue at a random point in time equilibrium and

$$\chi(z) = \frac{1+\frac{\theta}{\lambda}+\theta(1-\rho z)}{1+\frac{\theta}{\lambda}+\theta(1-\rho)}$$

the probability generating function of the additional number of customers at a random point in time equilibrium when the server is either idle or in glue period.

Thus, $P(z) = \prod_{M|M|1}(z)\chi(z)$ conforms that the decomposition law of [8] is also valid for our queueing system under steady-state.

There are several general descriptors of our queuing system, some of which are listed below:

Under steady-state, it can be seen that

$$P(server\ is\ busy) = \sum_{n=1}^{\infty} P_n = \frac{\frac{1+\frac{\theta}{\lambda}}{1-\rho}}{\theta + \frac{1+\frac{\theta}{\lambda}}{1-\rho}} \rho \tag{57}$$

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$$P(server\ is\ \ available) = \frac{\dfrac{1+\dfrac{\theta}{\lambda}}{1-\rho}}{\theta + \dfrac{1+\dfrac{\theta}{\lambda}}{1-\rho}} \qquad (58)$$

$$P(server\ is\ either\ idle\ or\ in\ glue\ \ period\ \ state) = P_0 + G = \frac{1+\dfrac{\theta}{\lambda}+\theta}{\theta + \dfrac{1+\dfrac{\theta}{\lambda}}{1-\rho}} \qquad (59)$$

and

$$P_w = P(An\, arriving\, customer\, has\, to\, wait\ \ for\, service) = 1 - P_0 - G = \frac{\theta + \dfrac{(1+\dfrac{\theta}{\lambda})\rho}{1-\rho}}{\theta + \dfrac{1+\dfrac{\theta}{\lambda}}{1-\rho}} \qquad (60)$$

The condition probabilities are

$$P(server\, is\, busy \mid server\, is\, available) = \rho \qquad (61)$$

and

$$P(server\, is\, idle \mid server\, is\, available) = 1-\rho \qquad (62)$$

The probability of atleast $k$ customers in the system is given as

$$P[X \geq k] = \sum_{n=k}^{\infty} P_n = \frac{\dfrac{1+\dfrac{\theta}{\lambda}}{1-\rho}}{\theta + \dfrac{1+\dfrac{\theta}{\lambda}}{1-\rho}}\rho^k \qquad (63)$$

It is to note that the conditional probabilities (61) and (62) depend only on the parameters $\lambda$ and $\mu$, and are independent of the other system parameters.

To obtain the mean number of customers in the system when the server is available, define the conditional generating function

$$\phi(z) = E[z^x \mid server\, is\, available] = \frac{1-\rho}{1-\rho z} \qquad (64)$$

So that the conditional expectation is

$$E[X \mid server\, is\, available] = \frac{\rho}{1-\rho} \qquad (65)$$

Evidently the results in equations (64) and (65) agree with the probability generating function of the system size and the mean system size of the $M \mid M \mid 1$ queueing model without conditioning the availability of the server.

The conditional probability generating function of the number of customers in the queue is defined as

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$$R(z) = E[z^Q \mid server\, is\, available] = [1+\rho(1-z)]\frac{1-\rho}{1-\rho z} \tag{66}$$

and its conditional expectation can be obtained as

$$E[Q \mid server\, is\ available] = \frac{\rho^2}{1-\rho} \tag{67}$$

For the sake of orientation, we introduce the following notations:

Let $U$ be the system throughput, which is the rate at which customers exit the queue; $\lambda_{eff}$, the effective arrival rate when the server is accessible; $E(W_s)$, the mean waiting time in the system; $E(W_q)$, the mean waiting time in the queue; $E(T)$, the mean cycle time of the system and $E(N)$, the expected number of customers served during the busy period. the following theorem summarizes the results:

**Theorem 6:**

For $\lambda < \mu$, and $\theta > 0$, under steady-state

$$U = \frac{\dfrac{(1+\frac{\theta}{\lambda})\lambda}{1-\rho}}{\theta + \dfrac{1+\frac{\theta}{\lambda}}{1-\rho}} \tag{68}$$

$$E(W_s) = \frac{1}{\mu-\lambda} \tag{69}$$

$$E(W_q) = \frac{\rho}{\mu(1-\rho)} \tag{70}$$

$$E[W_q \mid W_q > 0] = \frac{\theta + \dfrac{1+\frac{\theta}{\lambda}}{1-\rho}}{\theta + \dfrac{(1+\frac{\theta}{\lambda})\rho}{1-\rho}} \frac{\rho}{1-\rho}\frac{1}{\mu} \tag{71}$$

$$E(T) = \frac{\frac{1}{\theta}+\frac{1}{\lambda}}{1-\rho} \tag{72}$$

and

$$E(N) = \frac{1+\frac{\lambda}{\theta}}{1-\rho} \tag{73}$$

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**Proof:**

The system throughput, $U$, is the rate at which customers exit the queue whenever there are one or more customers in the system. With the exit rate $\mu$, we can obtain, after some algebra,

$$U = [1 - P_0 - G]\mu = \frac{(1+\frac{\theta}{\lambda})\mu}{\theta + \frac{1+\frac{\theta}{\lambda}}{1-\rho}} \tag{74}$$

where we have used equations (43) and (44)

The effective arrival rate $\lambda_{eff}$ (i.e the total arrival rate whenever the server is available) is defined as

$$\lambda_{eff} = \lambda G + \lambda P_0 + \lambda \sum_{n=1}^{\infty} P_n = \frac{\frac{(1+\frac{\theta}{\lambda})\lambda}{1-\rho}}{\theta + \frac{1+\frac{\theta}{\lambda}}{1-\rho}} \tag{75}$$

It is interesting to note from equations (74) and (75) that, under steady stae, the effective arrival rate $\lambda_{eff}$ is equal to the system throughput $U$, i.e. $\lambda_{eff} = U$, which verifies the well known classical Burke's theorem (1956) for our queueing system also.

Now, by using Little's law, the mean waiting time $E(W_s)$ (sojourn time) in the system is obtained from Equation (52) as

$$E(W_s) = \frac{E(X)}{\lambda_{eff}} = \frac{1}{\mu - \lambda} \tag{76}$$

In a similar way, the mean waiting time $E(W_q)$ in the queue is derived from (54) as

$$E(W_q) = \frac{E(Q)}{\lambda_{eff}} = \frac{\rho}{\mu(1-\rho)} \tag{77}$$

also

$$E[W_q \mid W_q > 0] = \frac{E(W_q)}{1-P-G} = \frac{\theta + \frac{(1+\frac{\theta}{\lambda})}{1-\rho}}{\theta + \frac{(1+\frac{\theta}{\lambda})\rho}{1-\rho}} \frac{\rho}{1-\rho} \tag{78}$$

Let $T$ denote the cycle length of the regenerative process of the queueing system under investigator. It is clear that the cycle length $T$ consists of a period during which the server is in glue period (GP), an idle period of the server (IP) and a busy period of the server (BP)

i.e $T = GP + IP + BP$

Moreover, the underlying queueing system being a Markov process, it can be seen that

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$$E(GP)=\frac{1}{\theta},\ E(IP)=\frac{1}{\lambda},\ E(BP)=\frac{\left(\frac{1}{\theta}+\frac{1}{\lambda}\right)\rho}{1-\rho}$$

Finally, $E(T)=E(GP)+E(IP)+E(BP)=\frac{1}{\theta}+\frac{1}{\lambda}+\frac{\left(\frac{1}{\theta}+\frac{1}{\lambda}\right)\rho}{1-\rho}$

$$E(T)=\frac{\frac{1}{\theta}+\frac{1}{\lambda}}{1-\rho} \tag{79}$$

$$E(T)=\frac{E(\tau)}{1-\rho},\quad \text{where}\quad E(\tau)=\frac{1}{\lambda}+\frac{1}{\theta}$$

More over the expected number of unblocked customers who enter the system during a cycle is

$$\lambda_{eff}E(T)=\frac{\frac{1}{\theta}+\frac{1}{\lambda}}{1-\rho}\quad\lambda \tag{80}$$

It is interesting to observe from equation (80) that

$$\lambda_{eff}E(T)=E(BP)\mu \tag{81}$$

The expected number $E(N)$ of unblocked customers served during the busy period of the system [22] is

$$E(N)=E(\tau)+E(BP)\lambda=\frac{\frac{1}{\lambda}+\frac{1}{\theta}\mu}{1-\rho} \tag{82}$$

From equations (80) - (82) we could extract a nice and interesting relation

$$\lambda_{eff}E(T)=E(BP)\mu=E(N) \tag{83}$$

## 6. Numerical Illustrations

In this section, we present some numerical examples using MATLAB in order to illustrate the effect of various parameters in the system performance measures. We study the effect of the system parameters on the folowing main performance measures of our queueing system:

- the probability $P_n(t)$ that there are $n$ customers in the system.
- the expected waiting time of customers in the system.
- the expected number of customers in the system.
- the glue period of the system.

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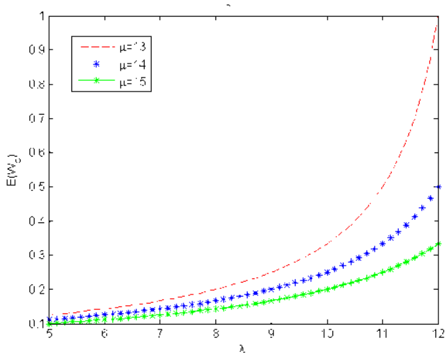


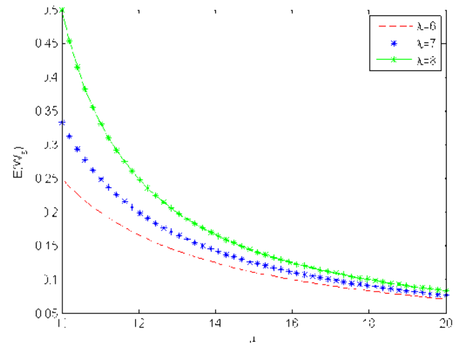
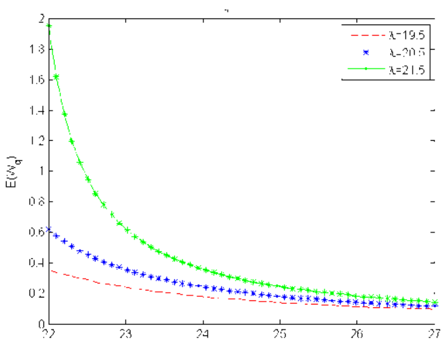


**Figure 1:** $E(W_S)$ **versus** $\lambda$ **Figure 2:** $E(W_S)$ **versus** $\mu$

Figure 1 shows that as the arrival rate increases the expected waiting time in the system increases for $\mu = 13$, $\mu = 14$, and $\mu = 15$. This is due to the fact that when the arrival rate ofthe customers is greater, the probability of the server being busy will increase as expected and hence will increase the expected waiting time. On the other hand Figure 2 shows that as the service rate increases the expected waiting time in the system decreases for a fixed value of $\lambda = 6$, $\lambda = 7$, and $\lambda = 8$. As the number of customers in the system increase it will increase the service rate and hence the increase in the expected waiting time.



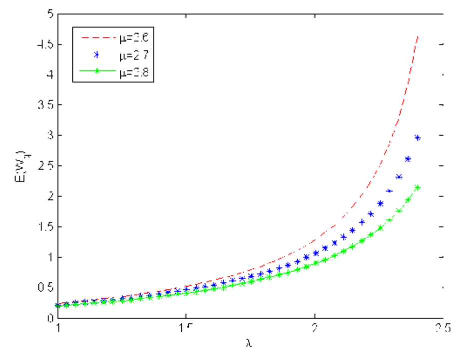
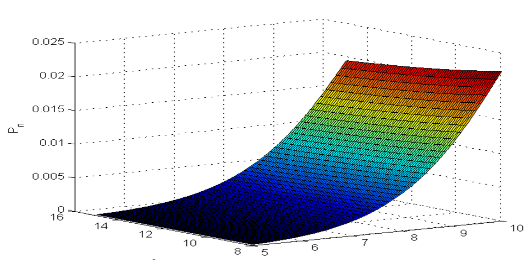


**Figure 3:** $E(W_q)$ **versus** $\mu$ **Figure 4:** $E(W_q)$ **versus** $\lambda$

For fixed values of $\lambda = 19.5$, $\lambda = 20.5$, and $\lambda = 21.5$, figure 3 shows that the expected waiting time in the queue decreases as the service rate increases. This is because if the service rate increases then the customers waiting time is decreases. The expected waiting time in the queue increases as the arrival rate increases for $\mu = 2.6$, $\mu = 2.7$, and $\mu = 2.8$ is shown in figure 4. This is since, supposing that the arrival rateexpands then the customers waiting time is expanded.



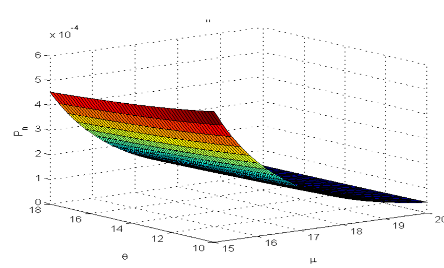


**Figure 5:** $P_n$ **with fixed** $\mu$ **Figure 6:** $P_n$ **with fixed** $\lambda$

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With different values of $\theta$ and $\mu$ and with fixed value of $\lambda = 8$, $n = 9$, figure 5 shows $P_n(t)$. As the service rate decreases the probability that there are n customers in the system decreases and hence the glue period decreases. Figure 6 shows $P_n(t)$ with fixed value of $\mu = 12$, $n = 8$ and varies with $\theta$ and $\lambda$. Here as the arrival rate increases the glue period increases and hence its probability increases.

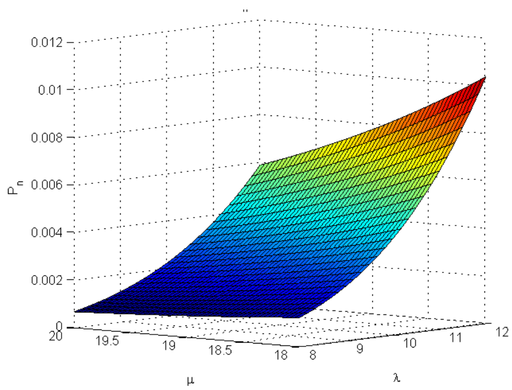


**Figure 7:** $P_n$ **with fixed** $\theta$

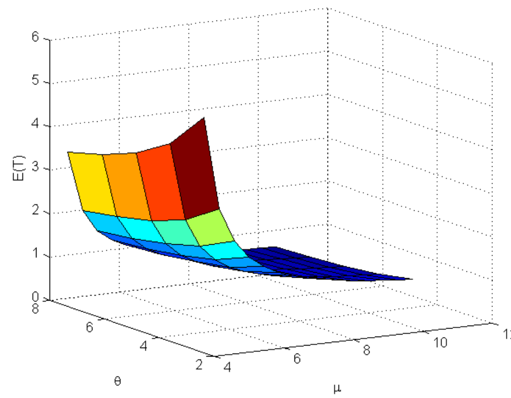


**Figure 8:** $E(T)$ **with fixed** $\lambda$

The value of $\lambda$ and $\mu$ fluctuates with fixed appraisal of $\theta = 10$, $n = 6$ in $P_n(t)$ is shown in Figure 7.With varied values of $\theta$ and $\mu$, figure 8 shows the expected time with fixed $\lambda = 4$.

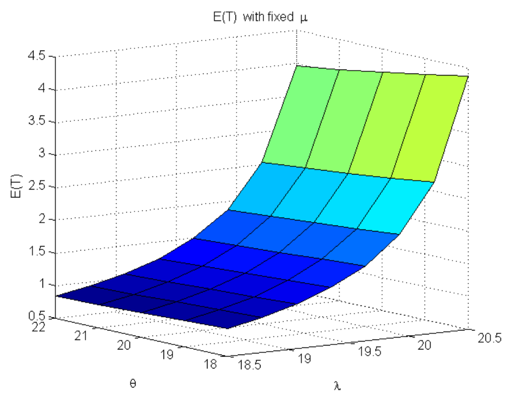


**Figure 9:** $E(T)$ **with fixed** $\mu$

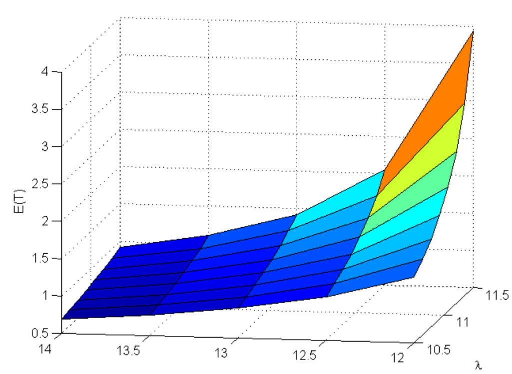


**Figure 10:** $E(T)$ **with fixed** $\theta$

The expected time increases as the service rate and glue period decrease. With fixed $\mu = 21$ and varying with $\theta$ and $\lambda$, Figure 9 shows the expected time increases. This is due to the fact that as the arrival rate increases the glue period increases and hence increase in mean waiting time. Figure 10 shows the expected time with fixed $\theta = 13$ and varies with $\mu$ and $\lambda$.

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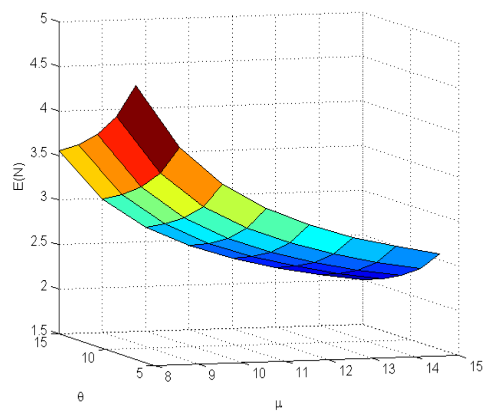


**Figure 11:** $E(N)$ **with fixed** $\lambda$

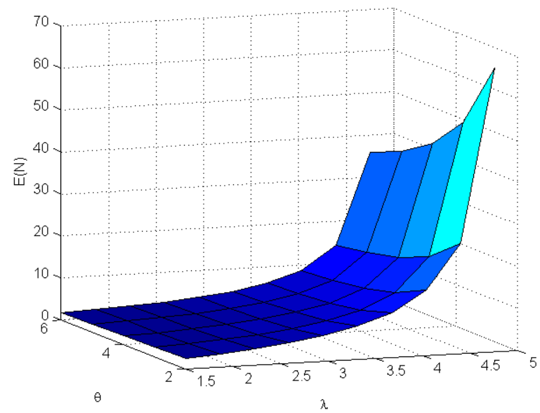


**Figure 12:** $E(N)$ **with fixed** $\mu$

The expected waiting time increases as service rate decreases and the arrival rate increases. With fixed $\lambda = 5$ and varying with $\theta$ and $\mu$, the expected number of customers can be seen in Figure 11. The expected number of customers increases as the service rate and the glue period decreases. Figure 12 shows the expected number of customers with fixed $\mu = 5$ and varies with $\theta$ and $\lambda$. As the glue period decreases and the arrival rate increases, the expected number of customers increases.

As $\mu$ and $\lambda$ varies the expected number of customers with fixed $\theta = 12$ is shown in Figure 13. This is because as the expected number of customers increases the service rate decreases and the arrival rate increases.

From the above numerical examples, we observed that the influence of parameters on the performance measures in the system and the results are coincident with the practical situations.

## 7. Conclusion

This paper addresses a fundamental issue in the existing queuing system, how to reduce the time individuals spend waiting in queues. This is a desirable goal because the waiting time is often assumed to be fruitless, even though this is not always the case. The derived results have numerous potential real-life applications, such as the Simple Mail Transfer Protocol, the well-known SMTP mail system for delivering messages to and from mail servers. Similarly, it can be implemented in the computer processing system and telephone consultation of medical service systems. This work can be further extended in many directions by incorporating the concepts of batch arrival, bulk service, and working breakdowns. This investigation is expected to be useful for for the system managers to make decisions regarding the system's size and other factors in a well-to-do manner.

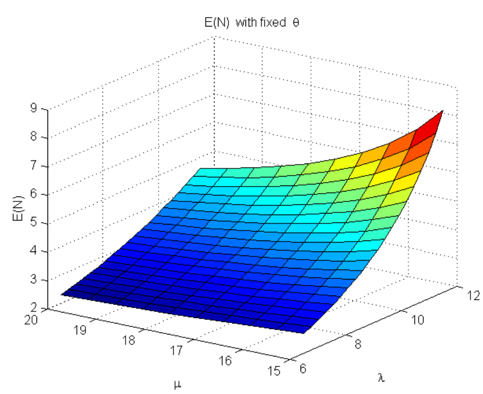


**Figure 13:** $E(N)$ **with fixed** $\theta$

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