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Neural Network Method For Solving A Nonlinear Problem Of Cross-Diffusion Task With Variable Density

Muhamediyeva Dildora Kabilovna, Madina Eldarovna Shaazizova,
Ilxom Tursunbayevich Ismailov, Malika Yuldashovna Doshchanova
Sultanmurat Uali uli Nasirov
E-mail: matematichka@inbox.ru
E-mail: madish_86@mail.ru
Tashkent University of Information Technologies named after Muhammad al-Khwarizmi, Tashkent, Uzbekistan.

Abstract: This paper examines the qualitative properties of solutions for cross-diffusion systems of a biological population with double nonlinearity and variable density. A self-similar approach is considered. A critical case is investigated.

Keywords: Self-similar equation, nonlinear task, critical case, interdiffusion coefficients, radially symmetric form.

Introduction

Formalization of the problem in the form of an objective function and a system is an integer nonlinear programming problem.

To solve such problems in general and the distribution problem in particular, various exact and approximate methods of combinatorial optimization are used. In most cases, the method that guarantees finding the optimal solution is a complete enumeration of all possible options. However, the set of options for feasible solutions to such problems grows rapidly with an increase in the dimension of the input data, which makes the use of the exhaustive search method unacceptable in practice.

Statement of the task

It was found that when considering such a problem, a critical case arises and the behavior of the solution in this case changes. In area Q={(t,x): 0< t < ∞, x∈ R^N }.

Let’s consider a cross-diffusion system of with double nonlinearity and variable density:

$$\begin{cases} \dfrac{\partial(\rho(x)u_1)}{\partial t} = div\left(D_1|x|^n u_2^{m_1-1}|\nabla u_1|^{p-2}\nabla u_1\right) + \rho(x)k_1u_1\left(1-u_1^{\beta_1}\right), \\ \dfrac{\partial(\rho(x)u_2)}{\partial t} = div\left(D_2|x|^n u_1^{m_2-1}|\nabla u_2|^{p-2}\nabla u_2\right) + \rho(x)k_2u_2\left(1-u_2^{\beta_2}\right), \end{cases} \tag{1}$$

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$$u_1\big|_{t=0} = u_{10}(x), \quad u_2\big|_{t=0} = u_{20}(x) \tag{2}$$

Here: $D_1|x|^n u_2^{m_1-1}|\nabla u_1|^{p-2}\nabla u_1, D_2|x|^n u_1^{m_2-1}|\nabla u_2|^{p-2}\nabla u_2$ - interdiffusion coefficients, $m_1, m_2, n, p, \beta_1, \beta_2, D_1, D_2$ - positive numeric parameters, $\nabla(.) - \underset{x}{grad}(.)$, $\beta_1, \beta_2 \geq 1$, $\rho(x) = |x|^{-l}$, $x \in R^N$ $l > 0$; $u_1 = u_1(t,x) \geq 0$, $u_2 = u_2(t,x) \geq 0$ - solution of a cross-diffusion system of a biological population with double nonlinearity and variable density.

We will study the properties of solutions to problem (1), (2) based on a self-similar analysis of solutions to the system of equations constructed by the method of nonlinear splitting and reference equations and by reducing system (1) to radially symmetric form.

Note that the change in (1) $u_1(t,x) = e^{-k_1 t} v_1(\tau(t), \varphi(x))$, $u_2(t,x) = e^{-k_2 t} v_2(\tau(t), \varphi(x))$ will bring it to the form:

$$\begin{cases} \dfrac{\partial(\rho(x)v_1)}{\partial\tau} = div\left(D_1|x|^n v_2^{m_1-1}|\nabla v_1|^{p-2}\nabla v_1\right) - \rho(x)k_1 e^{[(\beta_1-p+2)k_1-(m_1-1)k_2]t} v_1^{\beta_1+1}, \\ \dfrac{\partial(\rho(x)v_2)}{\partial\tau} = div\left(D_2|x|^n v_1^{m_2-1}|\nabla v_2|^{p-2}\nabla v_2\right) - \rho(x)k_2 e^{[(\beta_2-p+2)k_2+(m_2-1)k_1]t} v_2^{\beta_2+1}, \end{cases} \tag{3}$$

$$v_1\big|_{t=0} = v_{10}(x), \quad v_2\big|_{t=0} = v_{20}(x). \tag{4}$$

Let $k_1(p-(m_1+1)) = k_2(p-(m_2+1))$, then choosing

$$\tau(t) = \frac{e^{[(m_1-1)k_2+(p-2)k_1]t}}{(m_1-1)k_2+(p-2)k_1} = \frac{e^{[(m_2-1)k_1+(p-2)k_2]t}}{(m_2-1)k_1+(p-2)k_2}, \ \phi(x) = |x|^{p_1}/p_1, \ p_1 = (p-(n+l))/p,$$

we obtain the following system of cross-diffusion equations:

$$\begin{cases} \dfrac{\partial w_1}{\partial\tau} = \varphi^{1-s}\dfrac{\partial}{\partial\varphi}\left(\varphi^{s-1}D_1 w_2^{m_1-1}\left|\dfrac{\partial w_1}{\partial\varphi}\right|^{p-2}\dfrac{\partial w_1}{\partial\varphi}\right) - a_1\tau^{b_1}w_1^{\beta_1+1}, \\ \dfrac{\partial w_2}{\partial\tau} = \varphi^{1-s}\dfrac{\partial}{\partial\varphi}\left(\varphi^{s-1}D_2 w_1^{m_2-1}\left|\dfrac{\partial w_2}{\partial\varphi}\right|^{p-2}\dfrac{\partial w_2}{\partial\varphi}\right) - a_2\tau^{b_2}w_2^{\beta_2+1}. \end{cases} \tag{5}$$

Here:

$$a_1 = k_1\left((p-2)k_1+(m_1-1)k_2\right)^{b_1}, \qquad b_1 = \frac{(\beta_1-(p-2))k_1-(m_1-1)k_2}{(p-2)k_1+(m_1-1)k_2},$$

$$a_2 = k_2\left((m_2-1)k_1+(p-2)k_2\right)^{b_2}, \qquad b_2 = \frac{(\beta_2-(p-2))k_2-(m_2-1)k_1}{(m_2-1)k_1+(p-2)k_2},$$

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$$s = \frac{p(N-l)}{p-(n+l)},\ \ p \neq n+l\ .$$

When conditions are met $b_i = 0$, and $a_i(t) = const, i = 1,2$, cross-diffusion system has the form:

$$\begin{cases} \dfrac{\partial w_1}{\partial \tau} = \varphi^{1-s}\dfrac{\partial}{\partial \varphi}\left(\varphi^{s-1} D_1 w_2^{m_1-1}\left|\dfrac{\partial w_1}{\partial \varphi}\right|^{p-2}\dfrac{\partial w_1}{\partial \varphi}\right) - a_1 w_1^{\beta_1+1}, \\ \dfrac{\partial w_2}{\partial \tau} = \varphi^{1-s}\dfrac{\partial}{\partial \varphi}\left(\varphi^{s-1} D_2 w_1^{m_2-1}\left|\dfrac{\partial w_2}{\partial \varphi}\right|^{p-2}\dfrac{\partial w_2}{\partial \varphi}\right) - a_2 w_2^{\beta_2+1}, \end{cases} \tag{6}$$

Below we describe one of the ways to obtain a self-similar system for the system of equations (5). It consists of the following. Initially, we find a solution to the following system of equations:

$$\begin{cases} \dfrac{d\bar{w}_1}{d\tau} = -a_1 \bar{w}_1^{\beta_1+1}, \\ \dfrac{d\bar{w}_2}{d\tau} = -a_2 \bar{w}_2^{\beta_2+1}. \end{cases}$$

If $b_i = 0$, и $a_i(t) = const, i = 1,2$, then the solution of the equations has the following form:

$$\bar{w}_1(\tau) = (\tau(t))^{-\gamma_1},\ \gamma_1 = \frac{1}{\beta_1},\ \ \bar{w}_2(\tau) = (\tau(t))^{-\gamma_2},\ \gamma_2 = \frac{1}{\beta_2}.$$

In case $b_i \neq 0$, and $a_i(t) = const, i = 1,2$ we find a solution to the following system of equations:

$$\begin{cases} \dfrac{d\bar{w}_1}{d\tau} = -a_1 \tau^{b_1} \bar{w}_1^{\beta_1+1}, \\ \dfrac{d\bar{w}_2}{d\tau} = -a_2 \tau^{b_2} \bar{w}_2^{\beta_2+1}. \end{cases}$$

Solution of the equations is as follows:

$$\bar{w}_1(\tau) = (\tau(t))^{-\gamma_1},\ \gamma_1 = \frac{b_1+1}{\beta_1},\ \ \bar{w}_2(\tau) = (\tau(t))^{-\gamma_2},\ \gamma_2 = \frac{b_2+1}{\beta_2}.$$

Using the method of nonlinear splitting, the solution of system (5) is sought in the form:

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$$\begin{aligned} v_1(t,x) &= \bar{w}_1(\tau) z_1(\tau(t), \varphi(|x|)), \\ v_2(t,x) &= \bar{w}_2(\tau) z_2(\tau(t), \varphi(|x|)). \end{aligned} \tag{7}$$

If $\gamma_1(p-2)+\gamma_2(m_1-1)=\gamma_2(p-2)+\gamma_1(m_2-1)$, then parameter $\tau=\tau(t)$ is selected as follows:

$$\tau_1(\tau)=\int_0^\tau \bar{v}_1^{(p-2)}(t)\bar{v}_2^{(m_1-1)}(t)dt=\begin{cases} \dfrac{1}{1-[\gamma_1(p-2)+\gamma_2(m_1-1)]}(T+\tau)^{1-[\gamma_1(p-2)+\gamma_2(m_1-1)]}, & if\ \ 1-[\gamma_1(p-2)+\gamma_2(m_1-1)]\neq 0, \\ \ln(T+\tau), & if\ \ 1-[\gamma_1(p-2)+\gamma_2(m_1-1)]=0, \\ (T+\tau), & if\ \ p=2\ \ u\ \ m_1=1, \end{cases}$$

Then for the new variable $z_i(\tau, \varphi(|x|)),\ i=1,2$ we obtain the system of equations:

$$\begin{cases} \dfrac{\partial z_1}{\partial \tau} = \varphi^{1-s} div\left(\varphi^{s-1} D_1 z_2^{m_1-1} \left|\nabla z_1\right|^{p-2} \nabla z_1\right) + \psi_1 (z_1 - z_1^{\beta_1+1}), \\ \dfrac{\partial z_2}{\partial \tau} = \varphi^{1-s} div\left(\varphi^{s-1} D_2 z_1^{m_2-1} \left|\nabla z_2\right|^{p-2} \nabla z_2\right) + \psi_2 (z_2 - z_2^{\beta_2+1}), \end{cases} \tag{8}$$

where

$$\psi_1=\begin{cases} \dfrac{1}{(1-[\gamma_1(p-2)+\gamma_2(m_1-1)])\tau}, & if\ \ 1-[\gamma_1(p-2)+\gamma_2(m_1-1)>0, \\ \gamma_1 c_1^{-(1-[\gamma_1(p-2)+\gamma_2(m_1-1)])}, & if\ \ 1-[\gamma_1(p-2)+\gamma_2(m_1-1)=0, \end{cases}$$

$$\tag{9}$$

$$\psi_2=\begin{cases} \dfrac{1}{(1-[\gamma_2(p-2)+\gamma_1(m_2-1)])\tau}, & if\ \ 1-[\gamma_2(p-2)+\gamma_1(m_2-1)]>0, \\ \gamma_2 c_1^{-(1-[\gamma_2(p-2)+\gamma_1(m_2-1)])}, & if\ \ 1-[\gamma_2(p-2)+\gamma_1(m_2-1)]=0. \end{cases}$$

If $1-[\gamma_1(p-2)+\gamma_2(m_1-1)=0$, self-similar solution of system (9) has the form

$z_i(\tau(t),\varphi)= f_i(\xi),\ i=1,2,\ \xi=\varphi(|x|)/[\tau(t)]^{1/p}$. (10)

Then substituting (10) into (8) with respect to $f_i(\xi)$ we obtain the system of self-similar equations

$$\begin{cases} \xi^{1-s}\dfrac{d}{d\xi}\left(\xi^{s-1} f_2^{m_1-1}\left|\dfrac{df_1}{d\xi}\right|^{p-2}\dfrac{df_1}{d\xi}\right)+\dfrac{\xi}{p}\dfrac{df_1}{d\xi}+\mu_1 f_1(1-f_1^{\beta_1})=0, \\ \xi^{1-s}\dfrac{d}{d\xi}\left(\xi^{s-1} f_1^{m_2-1}\left|\dfrac{df_2}{d\xi}\right|^{p-2}\dfrac{df_2}{d\xi}\right)+\dfrac{\xi}{p}\dfrac{df_2}{d\xi}+\mu_2 f_2(1-f_2^{\beta_2})=0. \end{cases} \tag{11}$$

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where $\mu_1 = \dfrac{1}{(1-[\gamma_1(p-2)+\gamma_2(m_1-1)])}$ and $\mu_2 = \dfrac{1}{(1-[\gamma_2(p-2)+\gamma_1(m_2-1)])}$.

System (11) has an approximate solution of the form $\overline{f}_1 = A(a-\xi^{\gamma})^{n_1}$, $\gamma = p/(p-1)$,

$$\overline{f}_2 = B(a-\xi^{\gamma})^{n_2},$$

where A and B constant and

$$n_1 = \frac{(p-1)(p-(m_1+1))}{(p-2)^2-(m_1-1)(m_2-1)},\ n_2 = \frac{(p-1)(p-(m_2+1))}{(p-2)^2-(m_1-1)(m_2-1)}.$$

In this section, we solve the problem of choosing an initial approximation for the iterative process, which leads to a fast convergence to the solution of the Cauchy problem (1), (2), depending on the values of the numerical parameters and the initial data. For this purpose, the asymptotic representation of the solution found by us was used as an initial approximation.

Construction of the upper solution for cross-diffusion systems of a biological population

Let us start constructing an upper solution for system (11).

Note that the functions $\overline{f}_1(\xi)$, $\overline{f}_2(\xi)$ have properties:

$$\overline{f}_2^{\,m_1-1}\left|\frac{d\overline{f}_1}{d\xi}\right|^{p-2}\frac{d\overline{f}_1}{d\xi} = -A^{p-1}B^{m_2-1}\left(\gamma\gamma_1\right)^{p-1}\xi\,\overline{f}_1 \in C(0,\infty),$$

$$\overline{f}_1^{\,m_2-1}\left|\frac{d\overline{f}_2}{d\xi}\right|^{p-2}\frac{d\overline{f}_2}{d\xi} = -A^{m_2-1}B^{p-1}\left(\gamma\gamma_2\right)\ \xi\,\overline{f}_2 \in C(0,\infty)$$

and

$$\begin{cases} \xi^{1-s}\dfrac{d}{d\xi}\left(\xi^{s-1}\overline{f}_2^{\,m_1-1}\left|\dfrac{d\overline{f}_1}{d\xi}\right|^{p-2}\dfrac{d\overline{f}_1}{d\xi}\right) = -\left|\gamma\gamma_1\right|^{p-1}\gamma\gamma_1 A^{p-1}B^{m_1-1}\left(s\overline{f}_1+\xi\dfrac{d\overline{f}_1}{d\xi}\right), \\ \xi^{1-s}\dfrac{d}{d\xi}\left(\xi^{s-1}\overline{f}_1^{\,m_2-1}\left|\dfrac{df_2}{d\xi}\right|^{p-2}\dfrac{df_2}{d\xi}\right) = -\left|\gamma\gamma_2\right|^{p-1}\gamma\gamma_2 A^{m_2-1}B^{p-1}\left(s\overline{f}_2+\xi\dfrac{d\overline{f}_2}{d\xi}\right). \end{cases}$$

Let us choose A and B from the system of nonlinear algebraic equations

$$\left|\gamma\gamma_1\right|^{p-1}\gamma\gamma_1 A^{p-1}B^{m_1-1} = 1/p,$$
$$\left|\gamma\gamma_2\right|^{p-1}\gamma\gamma_2 A^{m_2-1}B^{p-1} = 1/p.$$

Then function $\overline{f}_1$, $\overline{f}_2$ are a Zeldovich-Kompaneets type solution for system (1) and in

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the domain $|\xi| < (a)^{(p-1)/p}$ they satisfy the system of equations

$$\begin{cases} \xi^{1-s}\dfrac{d}{d\xi}\left(\xi^{s-1}\overline{f}_2^{\,m_1-1}\left|\dfrac{d\overline{f}_1}{d\xi}\right|^{p-2}\dfrac{d\overline{f}_1}{d\xi}\right)+\dfrac{\xi}{p}\dfrac{d\overline{f}_1}{d\xi}+\dfrac{s}{p}\overline{f}_1=0, \\ \xi^{1-s}\dfrac{d}{d\xi}\left(\xi^{s-1}\overline{f}_1^{\,m_2-1}\left|\dfrac{d\overline{f}_2}{d\xi}\right|^{p-2}\dfrac{d\overline{f}_2}{d\xi}\right)+\dfrac{\xi}{p}\dfrac{d\overline{f}_2}{d\xi}+\dfrac{s}{p}\overline{f}_2=0 \end{cases}$$

in the classical sense.

Due to the fact that

$$\begin{cases} \xi^{s-1}\overline{f}_2^{\,m_1-1}\left|\dfrac{d\overline{f}_1}{d\xi}\right|^{p-2}\dfrac{d\overline{f}_1}{d\xi}=\xi^{s}\overline{f}_1, \\ \xi^{s-1}\overline{f}_1^{\,m_2-1}\left|\dfrac{d\overline{f}_2}{d\xi}\right|^{p-2}\dfrac{d\overline{f}_2}{d\xi}=\xi^{s}\overline{f}_2, \end{cases}$$

function $\overline{f}_1(\xi)$, $\overline{f}_2(\xi)$ and the flows have the following smoothness property

$$0\le \overline{f}_1(\xi),\ \xi^{s-1}\overline{f}_2^{\,m_1-1}\left|\dfrac{d\overline{f}_1}{d\xi}\right|^{p-2}\dfrac{d\overline{f}_1}{d\xi}=\xi^{s}\overline{f}_1\in C\left(0,\infty\right),$$

$$0\le \overline{f}_2(\xi),\ \xi^{s-1}\overline{f}_1^{\,m_2-1}\left|\dfrac{d\overline{f}_2}{d\xi}\right|^{p-2}\dfrac{d\overline{f}_2}{d\xi}=\xi^{s}\overline{f}_2\in C\left(0,\infty\right).$$

Let us choose A and B such that the inequalities

$$\begin{gathered} |\gamma\gamma_1|^{p-1}\gamma\gamma_1 A^{p-1}B^{m_1-1}\ge 1/p, \\ |\gamma\gamma_2|^{p-1}\gamma\gamma_2 A^{m_2-1}B^{p-1}\ge 1/p. \end{gathered} \tag{12}$$

Then, when

$$\begin{cases} \xi^{1-s}\dfrac{d}{d\xi}\left(\xi^{s-1}\overline{f}_2^{\,m_1-1}\left|\dfrac{d\overline{f}_1}{d\xi}\right|^{p-2}\dfrac{d\overline{f}_1}{d\xi}\right)+\dfrac{\xi}{p}\dfrac{d\overline{f}_1}{d\xi}=-\dfrac{s}{p}\overline{f}_1, \\ \xi^{1-s}\dfrac{d}{d\xi}\left(\xi^{s-1}\overline{f}_1^{\,m_2-1}\left|\dfrac{d\overline{f}_2}{d\xi}\right|^{p-2}\dfrac{d\overline{f}_2}{d\xi}\right)+\dfrac{\xi}{p}\dfrac{d\overline{f}_2}{d\xi}=-\dfrac{s}{p}\overline{f}_2, \end{cases}$$

then due to the fact that

$$\dfrac{d\overline{f}_1}{d\xi}\le 0,\ \dfrac{d\overline{f}_2}{d\xi}\le 0\ npu\ \xi\in(0,\infty),$$

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From (12) we have

$$\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1-}\overline{f}_2^{\,m_1-1}\left|\frac{d\overline{f}_1}{d\xi}\right|^{p-2}\frac{d\overline{f}_1}{d\xi}\right)+\frac{\xi}{p}\frac{d\overline{f}_1}{d\xi}\le 0,$$
$$\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}\overline{f}_1^{\,m_1-1}\left|\frac{d\overline{f}_2}{d\xi}\right|^{p-2}\frac{d\overline{f}_2}{d\xi}\right)+\frac{\xi}{p}\frac{d\overline{f}_2}{d\xi}\le 0,$$
$$\xi\in(0,\infty).$$

Theorem 1. If $u_i(0,x)\le u_{i\pm}(0,x),\ x\in R,$ $\overline{f}_1=A(a-\xi^{\gamma})^{n_1},\ \gamma=p/(p-1),$ $\overline{f}_2=B(a-\xi^{\gamma})^{n_2},\ n_1=\dfrac{(p-1)(p-(m_1+1))}{(p-2)^2-(m_1-1)(m_2-1)},\ n_2=\dfrac{(p-1)(p-(m_2+1))}{(p-2)^2-(m_1-1)(m_2-1)}$, then in the domain Q the solution of problem (1) satisfies the upper bound

$$u_1(t,x)\le u_{1+}(t,x)=e^{k_1t}\tau^{-\alpha_1}\overline{f}_1(\xi),\qquad \xi=\varphi(|x|)/[\tau(t)]^{1/p}.$$
$$u_2(t,x)\le u_{2+}(t,x)=e^{k_2t}\tau^{-\alpha_2}\overline{f}_2(\xi),$$

Note that the solution of system (1) for $\beta_i=\dfrac{(p-2)^2-(m_1-1)(m_2-1)}{(p-1)(p-(m_i+1))}$ has the following representation for

$$a=\left(P_1\gamma/B(\frac{1}{\gamma},1+n_1)\right)^{\frac{\gamma}{n_1}}=\left(P_2\gamma/B(\frac{1}{\gamma},1+n_2)\right)^{\frac{\gamma}{n_2}}.$$

where B(a,b)- Beta Euler function.

It is proved that this representation is the asymptotic behavior of self-similar solutions to systems (1).

$$\begin{cases}\tau^{-\frac{1}{\mu_1}}\displaystyle\int_{-\infty}^{\infty}(a-\xi_1^{\gamma})_+^{n_1}dx=P_1,\\ \tau^{-\frac{1}{\mu_2}}\displaystyle\int_{-\infty}^{\infty}(a-\xi_2^{\gamma})_+^{n_2}dx=P_2,\end{cases}$$

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$$\begin{cases} \tau^{-\frac{1}{\mu_1}} \int\limits_{-\infty}^{\infty} (a-\xi_1^{\gamma})_+^{n_1} dx = a^{\frac{n_1}{\gamma}} \dfrac{1}{\gamma} \int\limits_0^1 \eta^{\frac{1}{\gamma}-1} (1-\eta)^{n_1} d\eta = a^{\frac{n_1}{\gamma}} \dfrac{1}{\gamma} B(\dfrac{1}{\gamma}, 1+n_1) = P_1, \\ \tau^{-\frac{1}{\mu_2}} \int\limits_{-\infty}^{\infty} (a-\xi_2^{\gamma})_+^{n_2} dx = a^{\frac{n_2}{\gamma}} \dfrac{1}{\gamma} \int\limits_0^1 \eta^{\frac{1}{\gamma}-1} (1-\eta)^{n_2} d\eta = a^{\frac{n_2}{\gamma}} \dfrac{1}{\gamma} B(\dfrac{1}{\gamma}, 1+n_2) = P_2. \end{cases}$$

Here:

$$a = [P_1\gamma / B(\frac{1}{\gamma}, 1+n_1)]^{\frac{\gamma}{n_1}} = [P_2\gamma / B(\frac{1}{\gamma}, 1+n_2)]^{\frac{\gamma}{n_2}}.$$

At $n_1 > 0, n_2 > 0, n > 0$ we get the following functions

$$\bar{\theta}_1(\xi) = (a-\xi)_+^{n_1}, \ \bar{\theta}_2(\xi) = (a-\xi\ )_+^{n_2}.$$

Here: $a > 0$, $(y)_+ = \max(y, 0)$, $\xi < a$. For a global solution of the system of equations (1) to exist for the function $f_i(\xi), i = 1,2$ must satisfy the following inequality [5]:

$$\xi^{1-s} \frac{d}{d\xi}\left(\xi^{s-1-} \bar{f}_2^{m_1-1} \left| \frac{d\bar{f}_1}{d\xi} \right|^{p-2} \frac{d\bar{f}_1}{d\xi} \right) + \frac{\xi}{p} \frac{d\bar{f}_1}{d\xi} + \mu_1 f_1 (1 - f_1^{\beta_1}) \le 0,$$

$$\xi^{1-s} \frac{d}{d\xi}\left(\xi^{s-1} \bar{f}_1^{m_1-1} \left| \frac{d\bar{f}_2}{d\xi} \right|^{p-2} \frac{d\bar{f}_2}{d\xi} \right) + \frac{\xi}{p} \frac{d\bar{f}_2}{d\xi} + \mu_1 f_1 (1 - f_1^{\beta_1}) \le 0,$$

$$\xi \in (0, \infty).$$

Here:

$$\beta_1 = 1/n_2, \ \beta_2 = 1/n_1.$$

Development of an algorithm and program for solving cross-diffusion systems

Consider in the area $D = \Omega \times (0,T)$, $\Omega \subset R^N$, $\Omega = \{-b_\alpha < x_\alpha < b_\alpha, \quad \alpha = 1,2\}$ two-dimensional reaction problem with diffusion

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\left[u^\sigma \frac{\partial}{\partial x_1} u \right] + \frac{\partial}{\partial x_2}\left[u^\sigma \frac{\partial}{\partial x_2} u \right] - v_1(t) \frac{\partial u}{\partial x_1} - v_2(t) \frac{\partial u}{\partial x_2} + k(t) u \left(1 - u^\beta\right), \tag{17}$$

$$u = u(x_1, x_2, t), \quad |x| = \sqrt{(x_1)^2 + (x_2)^2},$$

$$x \in \Omega$$

with initial and boundary conditions

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$$u(0,x) = u0\left(x\right) \geq 0, \tag{18}$$

$$u\big|_{\Gamma} = \mu(x,t)\,,\;\; t \in (0,T)\,,\; \Gamma - \text{граница } \Omega \tag{19}$$

which, in the case of degenerating, is equivalent to the Cauchy problem with a finite initial function

$$u_0\left(t,x\right) = \bar{u}(t)\left(a - \frac{\delta}{4}\xi^2\right)^{1/\sigma};a=1; \qquad \xi = x/\tau^{1/2};\; \tau(t) = \int_0^t [\bar{u}(\eta)]^{\sigma}\,d\eta.$$

$$k(t,x) := k(t)\,;\quad k(t) = \frac{1}{(1+t)^{\alpha}}\,,\quad \alpha > 1;\; \bar{u}(t) = \left[\frac{1}{1+e^{-\int_0^t k(\eta)d\xi}}\right]^{1/\beta},$$

where

$$1)\;\tau(\infty) < +\infty\,,\quad 2)\; q\int e^{\int_0^t k(\eta)d\eta}\,d\eta < +\infty\,.$$

Let $0 < k(t,x) \in C(0,+\infty) \times R^N$. In Ω build a uniform mesh $\bar{\omega}_h$ by x_α, $(\alpha = 1,2)$ with steps $h_1 = \frac{b_1}{n_1}$ и $h_2 = \frac{b_2}{n_2}$:

$$\bar{\omega}_h = \left\{x_{ij} = \left(x_1^i, x_2^j\right),\quad x_1^i = ih_1,\quad x_2^j = jh_2,\quad i,j = 0,1,\ldots,n_\alpha,\quad \alpha = 1,2\right\},$$

and uniform grid in time $\bar{\omega}_\tau = \left\{t_k = k\tau,\quad \tau > 0,\quad k = 0,1,\ldots,m,\quad \tau m = T\right\},\; T > 0$.

The idea is as follows: Enter an intermediate value $\bar{y} = y^{k+1/2}$, где $y = y^k$, $\hat{y} = y^{k+1}$, k - layer number, which can be regarded as the value of y at $t{=}t_{k+1/2} = t_k + \tau/2$

$$\begin{cases} \dfrac{y^{k+1/2} - y^k}{0.5\cdot\tau} = \Lambda_1 y^{k+1/2} + \Lambda_2 y^k + q(y^k), \\ \dfrac{y^{k+1} - y^{k+1/2}}{0.5\cdot\tau} = \Lambda_1 y^{k+1/2} + \Lambda_2 y^{k+1} + q\left(y^{k+1}\right), \end{cases} \tag{20}$$

$$\Lambda_1 y^{k+1/2} = \frac{1}{h_1^2}\left[a_{i+1,j}\left(y^{k+1/2}\right)\left(y_{i+1,j}^{k+1/2} - y_{i,j}^{k+1/2}\right) - a_{i,j}\left(y^{k+1/2}\right)\left(y_{i,j}^{k+1/2} - y_{i-1,j}^{k+1/2}\right)\right] + \frac{v_{ij}^1}{h_1}\left(y_{i+1,j}^{k+1/2} - y_{i,j}^{k+1/2}\right),$$

$$\Lambda_2 y^k = \frac{1}{h_2^2}\left[b_{i,j+1}\left(y^k\right)\left(y_{i,j+1}^k - y_{i,j}^k\right) - b_{i,j}\left(y^k\right)\left(y_{i,j}^k - y_{i,j-1}^k\right)\right] + \frac{v_{ij}^2}{h_2}\left(y_{i,j+1}^k - y_{i,j}^k\right),$$

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$$q(y)=k(t,x_{1i},x_{2j})y_{i,j}(1-y_{i,j}^{\beta})\ ,\ |x|^m=\left(\sqrt{x_1^2+x_2^2}\right)^m .$$

Here the difference coefficients $a(y)$и $b(y)$ must satisfy the conditions of the second order of approximation and one of the following formulas is used to calculate:

$$\text{a)}\, a_{i,j}(y)=K\left(\frac{y_{i-1,j}+y_{i,j}}{2}\right),\ b_{i,j}(y)=K\left(\frac{y_{i,j-1}+y_{i,j}}{2}\right), i,j=1,2,...,n_\alpha-1,\quad \alpha=1,2\,, \tag{21}$$

$$\text{б)}\, a_{i,j}(y)=\frac{K\left(y_{i-1,j}\right)+K\left(y_{i,j}\right)}{2},\ b_{i,j}(y)=\frac{K(y_{i,j-1})+K(y_{i,j})}{2}, i,j=1,2,..,n_\alpha-1,\quad \alpha=1,2\,, \tag{22}$$

where $K(u)=u^{\sigma}$.

Using formula (22), we have

$$a_{i,j}(y)=\frac{1}{2}\left[(y_{i-1,j})^{\sigma}+(y_{i,j})^{\sigma}\right],\ b_{i,j}(y)=\frac{1}{2}\left[(y_{i,j-1})^{\sigma}+(y_{i,j})^{\sigma}\right].$$

We rewrite the initial and boundary conditions as follows:

$$\begin{cases} y_{i,j}^{0}=u_0(x), \qquad x\in\bar{\omega}_h, \\ y_{i,j}^{k+1}=\mu^{k+1}, \quad \text{при } j=0 \text{ и } j=n_2, \\ y_{i,j}^{k+1/2}=\bar{\mu}^{k+1/2}, \quad \text{при } \mathrm{i}=0 \text{ и } i=n_1, \end{cases} \tag{23}$$

where $\bar{\mu}=\frac{1}{2}\left(\mu^{k+1}+\mu^{k}\right)-\frac{\tau}{4}\Lambda_2\left(\mu^{k+1}-\mu^{k}\right)$, which is obtained from system (20), after eliminating the intermediate value $y^{k+1/2}$.

We rewrite (20) as:

$$\begin{cases} \dfrac{y_{i,j}^{k+1/2}}{0.5\cdot\tau}=\Lambda_1 y^{k+1/2}+F_{i,j}^{k}, \qquad & F_{i,j}^{k}=\dfrac{2}{\tau}y_{i,j}^{k}+\Lambda_2 y^{k}+q(y^{k}), \\ \dfrac{y_{i,j}^{k+1}}{0.5\cdot\tau}=\Lambda_2 y^{k+1}+\bar{F}_{i,j}^{k+1/2}, \qquad & \bar{\mathrm{F}}_{\mathrm{i,j}}^{k+1/2}=\dfrac{2}{\tau}y_{i,j}^{k+1/2}+\Lambda_1 y^{k+1/2}+q(y^{k+1/2}), \end{cases} \tag{24}$$

we also agree on the following notation: $y^{k}=y, \quad y^{k+1/2}=\bar{y}, \quad y^{k+1}=\hat{y}$.

To solve the resulting scheme of nonlinear equations, we use the iterative method.

$$\frac{\overset{s+1}{\bar{y}}_{i,j}}{0.5\cdot\tau}=\frac{1}{h_1^2}\left[a_{i+1,j}(\overset{s}{\bar{y}})\left(\overset{s+1}{\bar{y}}_{i+1,j}-\overset{s+1}{\bar{y}}_{i,j}\right)-a_{i,j}(\overset{s}{\bar{y}})\left(\overset{s+1}{\bar{y}}_{i,j}-\overset{s+1}{\bar{y}}_{i-1,j}\right)\right]+\frac{\nu_{ij}^{1}}{h_1}\left(y_{i+1,j}^{k+1/2}-y_{i,j}^{k+1/2}\right)+\overset{s}{F}_{i,j}=0, \tag{25}$$

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$$\frac{\overset{s+1}{\hat{y}}_{i,j}}{0.5\cdot\tau}=\frac{1}{h_1^2}\left[b_{i,j+1}(\overset{s}{\hat{y}})\left(\overset{s+1}{\hat{y}}_{i,j+1}-\overset{s+1}{\hat{y}}_{i,j}\right)-b_{i,j}(\overset{s}{\hat{y}})\left(\overset{s+1}{\hat{y}}_{i,j}-\overset{s+1}{\hat{y}}_{i,j-1}\right)\right]+\frac{\nu_{ij}^2}{h_2}\left(y_{i,j+1}^k-y_{i,j}^k\right)+\overset{s}{\bar{F}}_{i,j}=0, \tag{26}$$

where $a_{i,j}(y)$ and $b_{i,j}(y)$ are defined by formula (22).

he iterative process is performed according to the following schemes:

The approximation is performed by the Picard method (simple iteration):

$$\begin{cases}\dfrac{\overset{s+1}{\bar{y}}-y}{0.5\cdot\tau}=\Lambda_1\overset{s+1}{\bar{y}}+\Lambda_2 y+q(\overset{s}{\bar{y}}),\\ \dfrac{\overset{s+1}{\hat{y}}-\bar{y}}{0.5\cdot\tau}=\Lambda_1\bar{y}+\Lambda_2\overset{s+1}{\hat{y}}+q\left(\overset{s}{\hat{y}}\right).\end{cases}$$

In (25), introducing the notation

$$\overset{s}{A}_{i,j}=\frac{0.5\tau\cdot\left|x_{i+1,j}\right|^m}{h_1^2}\left(\overset{s}{\bar{y}}_{i,j}\right)^{\sigma}\cdot a_{i+1,j}\left(\overset{s}{\bar{y}}\right)+\frac{\tau}{h_1}\nu_{ij}^1,\quad \overset{s}{B}_{i,j}=\frac{0.5\tau\cdot\left|x_{i,j}\right|^m}{h_1^2}\left(\overset{s}{\bar{y}}_{i,j}\right)^{\sigma}\cdot a_{i,j}\left(\overset{s}{\bar{y}}\right),$$

$$\overset{s}{C}_{i,j}=\overset{s}{A}_{i,j}+\overset{s}{B}_{i,j}+1,\ s=0,1,2,...\ \ i,j=1,2,...,n_\alpha-1,\quad \alpha=1,2,$$

difference equation can be written as:

$$\begin{cases}\overset{s}{A}_{i,j}\overset{s+1}{\bar{y}}_{i+1,j}-\overset{s}{C}_{i,j}\overset{s+1}{\bar{y}}_{i,j}+\overset{s}{B}_{i,j}\overset{s+1}{\bar{y}}_{i-1,j}=-\overset{s}{F}_{i,j},\\ \bar{y}=\mu,\quad \text{при}\ \ i=0,n_1,\end{cases} \tag{27}$$

$i,j=1,2,...,n_\alpha-1,\quad \alpha=1,2.$

Accordingly, (26) can be written as:

$$\begin{cases}\overset{s}{\bar{A}}_{i,j}\overset{s+1}{\hat{y}}_{i+1,j}-\overset{s}{\bar{C}}_{i,j}\overset{s+1}{\hat{y}}_{i,j}+\overset{s}{\bar{B}}_{i,j}\overset{s+1}{\hat{y}}_{i,j-1}=-\overset{s}{\bar{F}}_{i,j},\\ \hat{y}=\bar{\mu},\quad \text{at}\ \ i=0,n_2,\end{cases} \tag{28}$$

where

$$\overset{s}{\bar{A}}_{i,j}=\frac{0.5\tau\cdot\left|x_{i,j+1}\right|^m}{h_2^2}\left(\overset{s}{\hat{y}}_{i,j}\right)^{\sigma}\cdot b_{i+1,j}\left(\overset{s}{\hat{y}}\right)+\frac{\tau}{h_2}\nu_{ij}^2,\quad \overset{s}{B}_{i,j}=\frac{0.5\tau\cdot\left|x_{i,j}\right|^m}{h_1^2}\left(\overset{s}{\bar{y}}_{i,j}\right)^{\sigma}\cdot a_{i,j}\left(\overset{s}{\bar{y}}\right),$$

$$\overset{s}{\bar{C}}_{i,j}=\overset{s}{\bar{A}}_{i,j}+\overset{s}{\bar{B}}_{i,j}+1,\ s=0,1,2,...\ \ i,j=1,2,...,n_\alpha-1,\quad \alpha=1,2.$$

For the numerical solution of problems (27) and (28), the sweep method is used.

System of equations (27) is solved along the lines $j = 1, 2, ..., n_2 - 1$ and is determined $\bar{y}$ at all grid points ω_h. Then the system of equations (28) is solved along the columns $i = 1, 2, ..., n_1 - 1$ defining $\hat{y}$ at all grid points ω_h. When passing from layer k + 1 to layer k + 2, the counting procedure is repeated. The results of the C ++ solution are shown in Fig.1.

Conclusion

The processes of multicomponent cross-diffusion systems of a biological population with double nonlinearity and variable density are simulated on a computer.

Estimates are obtained for solving the Cauchy problem for multicomponent cross-diffusion systems of a biological population with double nonlinearity and variable density.

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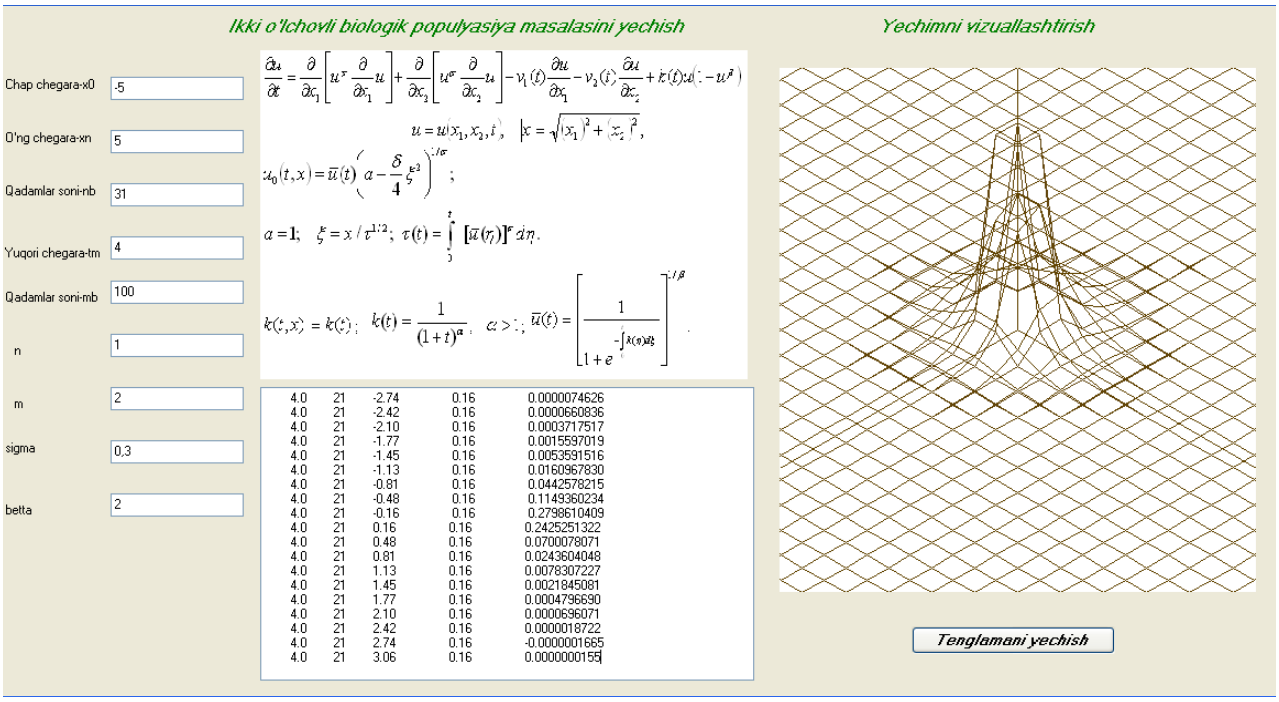
Solution results



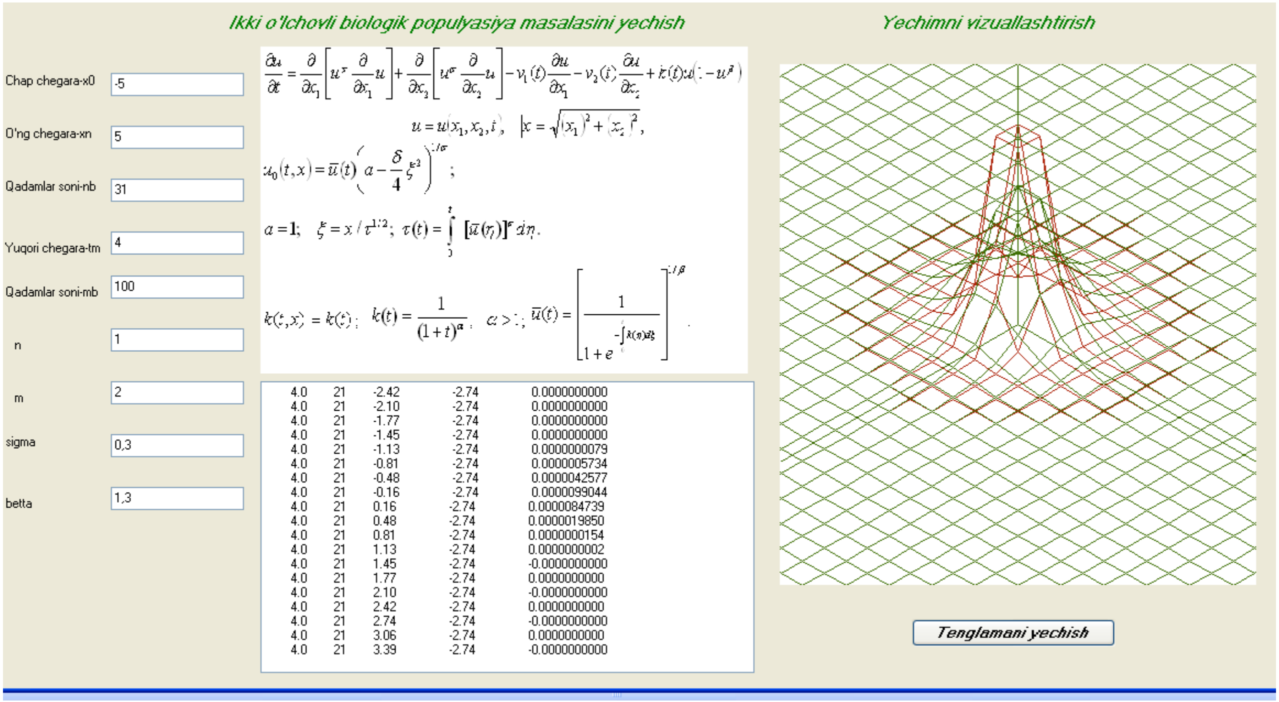


Figure 1. Solution results

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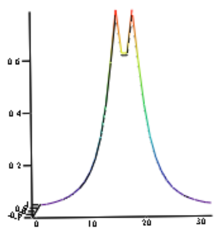
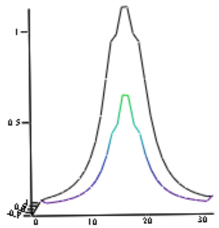
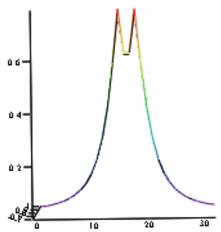
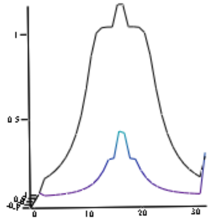
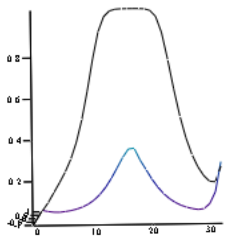
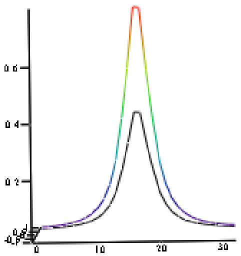
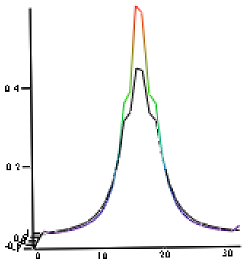
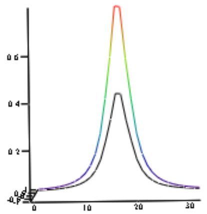
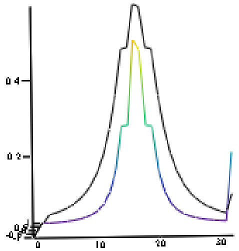
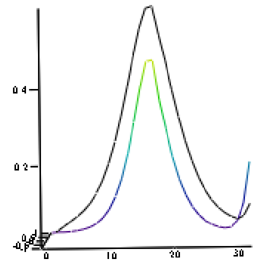
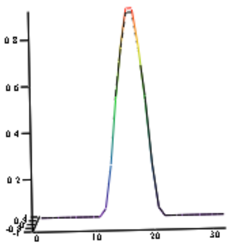
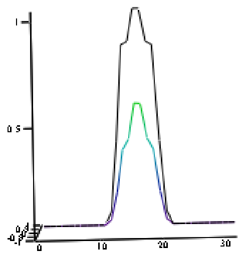
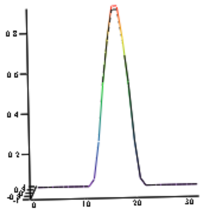
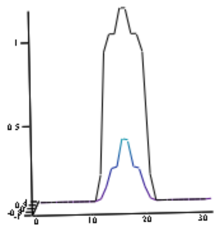
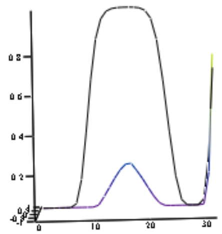
Parameter values	$t=1$ $\rho=\lvert x\rvert^{-l}$, $l=0$	$t=20$ $\rho=\lvert x\rvert^{-l}$, $l=0$	$t=1$ $\rho=\lvert x\rvert^{-l}$, $l=0.2$	$t=20$ $\rho=\lvert x\rvert^{-l}$, $l=0.2$	$t=20$ $\rho=\lvert x\rvert^{-l}$, $l=2.2$
$m_1=0.2$, $m_2=0.2$, $p=2.8$ $\beta_1=5$, $k_1=2$ $\beta_2=7$, $k_2=3$ $eps=10^{-3}$ $m_i+p-3=0$					
$m_1=0.2$, $m_2=0.2$, $p=2.8$ $\beta_1=0.5$, $k_1=2$ $\beta_2=0.7$, $k_2=3$ $eps=10^{-3}$ $m_i+p-3=0$					
$m_1=1.4$, $m_2=1.4$, $p=2.5$ $\beta_1=5$, $k_1=2$ $\beta_2=7$, $k_2=3$ $eps=10^{-3}$ $m_i+p-3>0$					

Figure 2. Results of a computational experiment in the one-dimensional case