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Estimates for Continuity Envelopes and Approximation Numbers of Generalized Bessel Potentials over Lorentz Space

Nesrin Hamadeh Alkhalil
Mathematical Institute named S. M. Nikolskii
Peoples' Friendship University of Russia
Moscow 117198, Miklukho-Maklaya str. 6, Russia
E-mail: nisreen.homadeh@gmail.

Abstract

In this paper we study spaces of Bessel potentials in n-dimensional Euclidean spaces. They are constructed on the basis of a rearrangement-invariant space (RIS) by using convolutions with Bessel– MacDonald kernels. The differential properties of potentials are characterized by their modulus of continuity of order k in the uniform norm. Specifically, the treatment covers spaces of Generalized Bessel potentials constructed over the basic weighted Lorentz space. In particular, we determine continuity envelope function. This result is then applied to estimate the approximation numbers of Generalized Bessel potentials when Generalized Bessel potentials constructed over the basic weighted Lorentz space.

Keywords: Rearrangement invariant space,Generalized Bessel, Continuity envelopes, modulus of continuity, Lorentz space, Approximation numbers.

Introduction

We study the space of Bessel potentials$H_E^G(\mathbb{R}^n)$ constructed by convolutions of functions with Bessel–MacDonald's kernels . Here the role of the basic space is played by a rearrangement-invariant space (RIS). The paper is organized as follows. Section 1- basic definitions of the potential theory. The main properties of kernels are considered and basic spaces for potentials are described. Section 2-contians some auxiliary results. Estimates for $\|u\|_C$are presented for potentials, and properties of moduli of continuity are discussed,determine continuity envelope function. The main results of the paper are presented in Section 3. In the Theorem 3.1 we determine continuity envelope function. This result is then applied to estimate the approximation numbers of Generalized Bessel potentials when Generalized Bessel potentials constructed over the basic weighted Lorentz space.

I.Basic definitions

The potential space $H_E^G \equiv H_E^G(\mathbb{R}^n)$ isdefined as the set of convolutions of potentials kernel with functions from the basic space.

$H_E^G(\mathbb{R}^n) = \{u = G * f: f \in E(\mathbb{R}^n)\}$.(1)

where $E(\mathbb{R}^n)$ is a rearrangement-invariant space (shortly: RIS). This uses the axiomatics introduced by C. Bennett and R. Sharpley [1]. We define

$\|u\|_{H_E^G} = \inf\{\|f\|_E : f \in E(\mathbb{R}^n), G * f = u\}$. (2)

The kernel of a representation G is called admissible if

$$G \in L_1(\mathbb{R}^n) + E'(\mathbb{R}^n).$$

Here the convolution $G * f$is defined as the integral

$(G * f)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} G(x - y)f(y)dy$.(3)

Moreover, let $E'(\mathbb{R}^n)$be the associated RIS, i.e. RIS with the norm:

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$$\|g\|_{E'} = \sup\left\{ \int_{\mathbb{R}^n} |fg| d\mu \ : f \in E, \|f\|_E \leq 1 \right\}. \qquad (4)$$

For the RIS $E(\mathbb{R}^n), E'(\mathbb{R}^n)$, we consider the spaces $\widetilde{E} = \widetilde{E}(\mathbb{R}_+)$, $\widetilde{E'} = \widetilde{E'}(\mathbb{R}_+)$ −their Luxemburg representations, i.e. RIS for which the following equalities are satisfied
$\|f\|_E = \|f^*\|_{\widetilde{E}}$, $\qquad \|g\|_{E'} = \|g^*\|_{\widetilde{E'}}$.
where $f^*$is the decreasing rearrangement of the function f, i.e. a nonnegative decreasing right continuous function on $\mathbb{R}_+ = (0, \infty)$, which is equi-measurable with f:

$$\mu_n \{x \in \mathbb{R}^n : |f(x)| > y\} = \mu_1\{t \in \mathbb{R}_+ : |f^*(t)| > y\}, \qquad y \in \mathbb{R}_+ . \qquad (5)$$

whereμ_n is n−dimensional Lebesgue measure.

The space of potentials $H_E^G \equiv H_E^G(\mathbb{R}^n)$isdefined as the set of convolutions of the kernels of potentials with functions from the basic space E.

For $R \in (0, \infty)$we introduce the class of monotone functions $\mathfrak{I}_n(R)$ as follows. the function$\Phi: (0, R) \to \mathbb{R}_+ \in \mathfrak{I}_n(R)$ belongs to $\mathfrak{I}_n(R)$, if Φsatisfies the following conditions:

1. decreasing and continuous at$(0, R)$,
2. there is a constant $c \in \mathbb{R}_+$, such that

$$\int_0^r \Phi(\rho)\rho^{n-1}\, d\rho \leq c\Phi(r)r^n, \quad r \in (0, R). \qquad (6)$$

The kernel of a representation $G \in L_1(\mathbb{R}^n)$, the properties of kernel are discussed in Definitions 1.1-1.3 below

Definition 1.1. Let Let$\Phi \in \mathfrak{I}_n(R)$. We assume that $G \in S_R(\Phi)$, if

$$G(x) \cong \Phi(|x|), \ 0 < \rho = |x| < R, \ R \in \mathbb{R}_+ .$$

Definition 1.2. Let $\Phi \in \mathfrak{I}_n(R)$. We assume that $G \in S_R(\Phi; X)$, where $X(\mathbb{R}^n)$ is an RIS, if

$$G(x) = G_R^0(x) + G_R^1(x) ;$$

$B_R = \{x \in \mathbb{R}^n : |x| < R\}, R \in \mathbb{R}_+$.
$G_R^0(x) = G(x)\chi_{B_R}(x)$; $\ G_R^1(x) = G(x)\chi_{B_R^c}(x)$,
$G_R^0(x) \cong \Phi(|x|)$, at $|x| < R$, $\ G_R^1(x) \in X(\mathbb{R}^n)$.

Definition 1.3. The potentials $u \in H_E^G(\mathbb{R}^n)$ are called generalized Bessel potentials, if
$\Phi \in \mathfrak{I}_n(R)$, where $R \in \mathbb{R}_+$.
$G \in S_R(\Phi ; L_1 \cap E')$, $\qquad \int_{\mathbb{R}^n} G\, dx \neq 0$.

Remark 1.4. If $\mathbf{G}^* \in \widetilde{E'}(0, T)$ and $u = G * f$: $f \in E(\mathbb{R}^n)$ then $u \in C(\mathbb{R}^n)$.

Definition 1.5. Modulus of continuity for $u \in C(\mathbb{R}^n)$in the uniform norm is defined as:

$$\omega_C^\kappa(u ; \tau) = \sup\left\{\left\|\Delta_h^k u\right\|_c : |h| \leq \tau\right\}, \qquad \tau \in \mathbb{R}_+ . \qquad (7)$$

Definition 1.6. [4]For a (quasi-) normed function space X on $\mathbb{R}^n$ with $X \to C$, its continuity envelope function $\xi_{C,k}^X(t) : (0, \infty) \to [0, \infty]$ is defined by

$$\xi_{C,k}^X(t) = \sup_{\|f\|_X \leq 1} \frac{\omega_C^\kappa(f ; t)}{t^k}, t > 0. \qquad (8)$$

Definition 1.7. [4]We can also define the majorant function

$$e_k^X(t) := t^k \xi_{C,k}^X(t) = \sup_{\|f\|_X \leq 1} \omega_C^\kappa(f ; t), \ t \geq 0, \ (9)$$

$e_k^X$is a non-negative, monotonically increasing function. Moreover, one can also consider some envelope function adapted to higher-order smoothness moduli,

$$\xi_{C,k}^X(t) = \sup_{\|f\|_X \leq 1} \frac{\omega_C^\kappa(f ; t)}{t^k} = t^k e_k^X(t), \quad t \geq 0, \quad k \in \mathbb{N}.$$

In particular we denote $\xi^{X}_{C,1} = \xi^{X}_{C}$.

we want to focus on the relation between continuity envelopes and approximation numbers of compact embeddings. We briefly recall this concept.

Definition 1.8.[4] Let A1 and A2be two complex (quasi-) Banach spaces and let $T \in L(A1, A2)$ be a linear and continuous operator from A1 into A2. The k-th approximation number of T is given by

$a_m(T) = \inf\{\| T - S \|: S \in L(A1, A2), \mathrm{rank}\, S < m.$(10)

Now let $\Omega \subset \mathbb{R}^n$be some bounded domain, X some function space on$\mathbb{R}^n$, and $X(\Omega)$ be defined by restriction. Assume that $X \hookrightarrow C(\Omega)$. It was proved in [8] that there exists $c > 0$ such that for all $m \in \mathbb{N}$,

$a_{m+1}(\mathrm{id}: X(\Omega) \hookrightarrow C(\Omega)) \leq cm^{-\frac{1}{n}} \xi^{X}_{C,k}\left(m^{-\frac{1}{n}}\right)$,

Definition 1.9. The Lorentz spaces$\Lambda^q(v)$, where v> 0, is measurable function, is the space of measurable functions with finite (quasi) norms:

$$\|f\|_{\Lambda^q(v)} = \begin{cases} (\int_0^\infty f^{*q}(t)v(t)dt)^{\frac{1}{q}} & ; 1 \leq q < \infty, \\ \text{ess } \sup_{t \in (0,\infty)}\{f^*(t)v(t)\} & ; q = \infty. \end{cases} \quad (11)$$

II. Auxiliary theorems

Theorem 2.1[7] Let $G \in L_1(\mathbb{R}^n), G \neq 0, \phi(\tau) = G^*(\tau), \tau \in \mathbb{R}_+$, and function $f: \mathbb{R}^n \longrightarrow \mathbb{R}_+$, is such that with some T$\in \mathbb{R}_+$.

$\int_0^T \phi(\tau) f^*(\tau) d\tau < \infty$.

1. For convolution$u(x) = \int_{\mathbb{R}^n} G(x-y) f(y) dy, \quad x \in \mathbb{R}^n$,

the following estimate holds$sup_{x \in \mathbb{R}^n} |u(x)| c_0 \int_0^T \phi(\tau) f^*(\tau) d\tau$,

$$c_0 = 1 + \left(\int_T^\infty \phi(\tau)\, d\tau\right)\left(\int_0^T \phi(\tau)\, d\tau\right)^{-1}.$$

2. Let more$G \in G^k(\mathbb{R}^n \backslash \{0\}), k \in \mathbb{N}$,and for$G_k(x) := \sum_{|\alpha|=k} |D^\alpha G(x)|$, $\quad x \in \mathbb{R}^n$,

with $c_1 \in \mathbb{R}_+$estimate takes place

$$|G_k(x)| \leq c_1 \psi_k(|x|), x \in \mathbb{R}^n, (*)$$

where

$0 \leq \psi_k(\tau) := \Psi_k\left(\left(\frac{\tau}{V_n}\right)^{\frac{1}{n}}\right) \downarrow$on $\mathbb{R}_+$,

and correlations hold$\psi_k(\tau) \leq \tau^{-k\backslash n} \phi(\tau), \tau \in (0, T], \int_T^\infty \psi_K(\tau)\, d\tau < \infty$.

Then the convolution, is continuous on$\mathbb{R}_+$ and for $t \in (0, T]$

$$\omega_C^K\left(u\,;\, t^{\frac{1}{n}}\right) \leq c_2 \int_0^T \left[\frac{\tau^{-\frac{\kappa}{n}}}{\tau^{-\frac{\kappa}{n}} + t^{-\frac{\kappa}{n}}}\right] \phi(\tau) f^*(\tau) d\tau. \quad (**)$$

Here $c_2 = c_1 \tilde{c} d$,, where

$$d = 1 + \frac{2}{T\psi_K(T)}\left(\int_T^\infty \psi_K(\tau)\, d\tau\right),$$

c_1 −constant of condition $(*)$, $\tilde{c} = \tilde{c}(k, n) \in \mathbb{R}_+$.

Under the condition $\phi(\tau) \in \tilde{E}'((0, T)$ inequality (3.1) performed for any function $f \in E(\mathbb{R}^n)$, that

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is Theorem 3.1 is applicable for any potential u $\in H_E^G(\mathbb{R}^n)$, since the formula$u(x) = \int_{\mathbb{R}^n} G(x - y)f(y)dy,\ \ x \in \mathbb{R}^n,$ is true for it.

Lemma 2.1.[7] Let the following condition be satisfied:$\int_t^\tau \tau^{-\frac{\kappa}{n}}\phi(\tau)d\tau \le B_0\, t^{1-\frac{\kappa}{n}}\phi(t)\ \ , t \in (0,T]$, where $B_0 \in \mathbb{R}_+$ independent of t. In addition, let the conditions of Theorem 2.1 be fulfilled. Then $\omega_C^K\left(u\,;\ t^{\frac{1}{n}}\right) \le c_2 \int_0^t \phi(\tau) f^*(\tau)d\tau\ \ , t \in (0,T]$, where $c_3 = (1 + B_0)c_2$, c_2 – constant from $(**)$.

Theorem 2.2. [3] Let $E = \Lambda^p(v)$ be the Lorentz space. Here weassume that the following condition is satisfied

$$\int_t^\tau \tau^{-\frac{\kappa}{n}}\phi(\tau)d\tau \le B_0\, t^{1-\frac{\kappa}{n}}\phi(t)\ ,\ \ t \in (0,T],$$

where $B_0 \in \mathbb{R}_+$is independent of t. Is satisfied. So, if $\phi = G^* \in \tilde{E}'((0,T)$ and $u = G * f\colon f \in \Lambda^p(v)$ there are the following:

1) $u \in C(\mathbb{R}^n)$.

2)Let $0 < p \le 1, then$

$$\omega_C^k\left(u\,;\ t^{\frac{1}{n}}\right) \le c_3 \breve{w}_0\left(t^{\frac{1}{n}}\right)\|u\|_{H^G_{\Lambda^p(v)}}, \qquad (12)$$

where $\breve{w}_0\left(t^{\frac{1}{n}}\right) = \sup_{\zeta\in(0,t)}\left\{\frac{\int_0^\zeta \varphi\ (\tau)d\tau}{\left(\int_0^\zeta \nu(t)dt\right)^{\frac{1}{p}}}\right\}$.

3) Let $1 < p < \infty$, then

$$\omega_C^K\left(u\,;\ t^{\frac{1}{n}}\right) \le c_5 A_k(t)\|u\|_{H^G_{\Lambda^p(v)}}, \qquad (13)$$

where

$$A_k(t) \approx \left[\int_0^t\left[\left(\int_0^\zeta \varphi\, d\tau\right)\left(\int_0^\zeta \upsilon d\tau\right)^{\frac{-1}{p}}\right]^{\acute{p}} \frac{\upsilon(\zeta)d\zeta}{\int_0^\zeta \upsilon d\tau} + \left(\int_0^\zeta \varphi\, d\tau\right)^{\acute{p}}\left(\int_0^\zeta \upsilon d\tau\right)^{\frac{-\acute{p}}{p}}\right]^{\frac{1}{\acute{p}}},$$

where $\acute{p} = \frac{p}{p-1}$.

III. Formulation of the results

Theorem 3.1. Let $E = \Lambda^p(v)$ be the Lorentz space. Here we preserve the above notations and assume that the conditions of Theorem 2.2are satisfied. So, there are the following estimates.

Let $0 < p \le 1, then$

$$\xi_{C,k}^X(t) \le 2c_0 \frac{\breve{w}_0\left(t^{\frac{1}{n}}\right)}{t^{k}/_{n}}, \qquad \text{where } \breve{w}_0\left(t^{\frac{1}{n}}\right) = \sup_{\zeta\in(0,t)}\left\{\frac{\int_0^\zeta \varphi\ (\tau)d\tau}{\left(\int_0^\zeta \nu(t)dt\right)^{\frac{1}{p}}}\right\}.$$

Let $1 < p < \infty$, then$\xi_{C,k}^X(t) \le 2c_1 \frac{A_k(t)}{t^{k}/_{n}}$,

$$A_k(t) \approx \left[\int_0^t\left[\left(\int_0^\zeta \varphi\, d\tau\right)\left(\int_0^\zeta \upsilon d\tau\right)^{\frac{-1}{p}}\right]^{\acute{p}} \frac{\upsilon(\zeta)d\zeta}{\int_0^\zeta \upsilon d\tau} + \left(\int_0^\zeta \varphi\, d\tau\right)^{\acute{p}}\left(\int_0^\zeta \upsilon d\tau\right)^{\frac{-\acute{p}}{p}}\right]^{\frac{1}{\acute{p}}},$$

where $\acute{p} = \frac{p}{p-1}$.

Proof:

Let $0 < p \leq 1$, from inequality (12) we can write

$$\frac{\omega_C^k\left(u\,;\, t^{\frac{1}{n}}\right)}{t^{\frac{k}{n}}} \leq c_3 \frac{\breve{w}_0\left(t^{\frac{1}{n}}\right)}{t^{\frac{k}{n}}} \|u\|_{H_E^G}.$$

Then we put in inequality (8)

$$\xi_{C,k}^{H_E^G}\left(t^{\frac{1}{n}}\right) = \sup_{\|u\|_{H_E^G} \leq 1} \frac{\omega_C^k\left(u\,;\, t^{\frac{1}{n}}\right)}{t^{\frac{k}{n}}} \leq c_3 \frac{\breve{w}_0\left(t^{\frac{1}{n}}\right)}{t^{\frac{k}{n}}}.$$

Moreover, we put here formula (9), and obtain

$$e_k^{H_E^G}\left(t^{\frac{1}{n}}\right) := t^{\frac{k}{n}} \xi_C^X(t) \leq c_3 \breve{w}_0\left(t^{\frac{1}{n}}\right),$$

Let $1 < p < \infty$, from inequality (13) we can write

$$\frac{\omega_C^K\left(u\,;\, t^{\frac{1}{n}}\right)}{t^{\frac{k}{n}}} \leq c_5 \frac{A_k(t)}{t^{\frac{k}{n}}} \|u\|_{H_{\Lambda^p(v)}^G},$$

Then we put in inequality (8)

$$\xi_{C,k}^{H_E^G}\left(t^{\frac{1}{n}}\right) = \sup_{\|u\|_{H_E^G} \leq 1} \frac{\omega_C^K\left(u\,;\, t^{\frac{1}{n}}\right)}{t^{\frac{k}{n}}} \leq c_5 \frac{A_k(t)}{t^{\frac{k}{n}}} \quad ,$$

Moreover, we put here formula (9), and obtain

$$e_k^{H_E^G}\left(t^{\frac{1}{n}}\right) := t^{\frac{1}{n}} \xi_C^X\left(t^{\frac{1}{n}}\right) \leq c_5 A_k(t),$$

IV. Conclusions

Let $E = \Lambda^p(v)$ be the Lorentz space. Here we preserve the above notations and assume that the conditions of Theorem 3.1 are satisfied. So, there are the following.

Let $\Omega \subset \mathbb{R}^n$ be some bounded domain, $H_E^G(\Omega)$ be defined by restriction. We have $H_E^G(\Omega) \hookrightarrow C(\Omega)$. We proved in that there exists $c > 0$ such that for all $m \in \mathbb{N}$,

if $0 < p \leq 1$, then $a_{m+1}\left(id : H_E^G(\Omega) \hookrightarrow C(\Omega)\right) \leq cm^{-\frac{1}{n}} \xi_{C,k}^X\left(m^{-\frac{1}{n}}\right) \leq c_3\, c\breve{w}_0\left(m^{-\frac{1}{n}}\right)$;

ift $1 < p < \infty$, then $a_{m+1}\left(id : H_E^G(\Omega) \hookrightarrow C(\Omega)\right) \leq cm^{-\frac{1}{n}} \xi_{C,k}^X\left(m^{-\frac{1}{n}}\right) \leq cc_5 A_k\left(m^{-\frac{1}{n}}\right)$,

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