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The Modular Inequalities for Hardy-Type Operators on Monotone Functions in Orlicz Space

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ABSTRACT

The purpose of this paper is to study the behaviour of integral operators of Hardy-type on monotone functionin orlicz spacewith general weight.on weighted Orlicz spaces. The result is based on the theorem on the reduction of modular inequalities for positively homogeneous operators on the cone Ω, which enables passing to modular inequalities for modified operators on the cone of all nonnegative functions from an Orlicz space. It is shown that, for the Hardy operator, the modified operator is a generalized Hardy-type operator. This enables us to establish explicit criteria for the validity of modular inequalities.

Key words: modular inequalities, norm inequalities, Orlicz space, cone of decreasing functions, positively homogeneous operators

I. INTRODUCTION

In this paper, we consider modular inequalities for Hardy-type operators on the cone Ω of positive decreasing functions from weighted Orlicz spaces. We use a general theorem (proved in [1]) on the reduction of modular inequalities for positively homogeneous operators on the cone Ω, which enables passing to modular inequalities for modified operators on the cone of all nonnegative functions from Orlicz space. It is based on the duality theorem describing the associated norm for the cone Ω. We follow, mostly, the notation used in the book [2, Sec. 8, Chap. 4] of Bennett and Sharpley. In the paper, we concretize modular inequalities for the case in which the positive operator is a Hardy-type operator. It is shown that, in that case, the modified operator is a generalized Hardy operator in the Jim Qile Sun notation [1]. This allows us to use an approach developed in [4], as well as its generalization and modification obtained by Jim Quile Sun [1], [5], to establish of explicit criteria for the validity of modular inequalities.

Definition 1. Function $\Phi:[0,+\infty)\to[0,+\infty)$ is called a Young's function if it satisfies conditions: $\Phi(0)=0, \lim_{x\to\infty}\Phi(x)=+\infty$.
An N-function Φ is continuous Young's function such that
$\Phi(t)=\int_0^t \phi$,
whereϕ is a nondecreasing, right continuous function defined on $[0,+\infty)$ with $\phi(0)=0, \phi(+\infty)=+\infty$. Let$\phi^{-1}$ be the right continuous inverse function of ϕ, and define
$\Psi(t)=\int_0^t \phi^{-1}$.
$\Psi$is called the complementary function of Φ.

Definition 2.a) An N-function Φ is said to satisfy the Δ_2 condition (we write $\Phi\in\Delta_2$)if there is a constant $B > 0$, such that

$$\Phi(2t)\le B\Phi(t), \forall t>0. \qquad (1)$$

b) We write $\Phi_1 \prec\prec \Phi_2$ if there is a constant $L_0>0$, such that inequality

$$\sum_i \Phi_2\circ\Phi_1^{-1}(a_i)\le L_0\,\Phi_2\circ\Phi_1^{-1}\left(\sum_i a_i\right), \qquad (2)$$

holds for every sequence $\{a_i\}$ with $a_i\ge 0$.
c) Let v be a positive, measurable weight function and Φ be an N-function. The Orlicz space$L_{\Phi,v}$ consists of all measurable function f (modulo equivalence almost everywhere) with

$$\|f\|_{\Phi,v}=\inf\left\{\lambda>0, \int_0^\infty \Phi(\lambda^{-1}|f(x)|)v(x)\,dx\le 1\right\}<\infty. \qquad (3)$$

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We call $\|.\|_{\Phi,v}$ the Luxemburg norm.

The Orlicz norm of a function f is given by

$$\|f\|'_{\Psi,v} = \sup\left\{\int_0^\infty |fg| v dx : \int_0^\infty \Psi(|g|) v dx \le 1\right\} \qquad (4)$$

Remark 1. $L_{\Phi,v}$ is a Banach space and the Luxemburg and Orlicz norms are equivalent . In fact,

$\|f\|_{\Phi,v} \le \|f\|'_{\Psi,v} \le 2\|f\|_{\Phi,v}$.

Consider the cone of nonnegative decreasing functions from the Orlicz space:

$$\Omega = \{f \in L_{\Phi,v} : 0 \le f \downarrow\}. \qquad (5)$$

For $g \in M_+$, we introduce the following associated norm on the cone Ω :

$$\|g\|'_\Omega = \sup\left\{\int_0^\infty f\, g\, dt : \quad f \in \Omega ;\ \|f\|_{\Phi,v} \le 1\right\} \qquad (6)$$

Proposition 1 ([4]). Let Φ, Ψ be the complementary Young functions, the Young function Φ satisfies Δ_2-condition, let $v \in M_+$, and let

$$0 < V(t) := \int_0^t v d\tau < \infty, \quad t \in R_+, \quad V(+\infty) = +\infty. \qquad (7)$$

The following two-sided estimate holds:

$$\|g\|'_\Omega \cong \|\Re_0(g)\|_{\Psi,v} = \inf\left\{\lambda > 0 : \int_0^\infty \Psi\left(\lambda^{-1} |\Re_0(g;t)|\right) v(t) dt \le 1\right\}, \qquad (8)$$

where

$$\Re_0(g;t) := V(t)^{-1} \int_0^t g(\tau)\, d\tau,\ t \in R_+. \qquad (9)$$

Here and below we use the notation

$$A \cong B \iff \exists c = c(a) \in [1,\infty): \quad c^{-1} \le A/B \le c. \qquad (10)$$

In the following considerations, we will use the formula for the conjugate operator to the operator:

$$\Re_0^*(f;\tau) = \int_\tau^\infty \frac{f(t)}{V(t)}\, dt,\ \tau \in R_+. \qquad (11)$$

Let us now state the main result of this section allowing us to reduce modular inequalities for operators on the cone Ω to modular inequalities for modified operators on the cone M_+ .

Proposition 2 ([10]). Let T and T^* be positively homogeneous operators that take M_+ to M_+ and are adjoint, i.e.,

$$\int_{R_+} gTf\, d\tau = \int_{R_+} f\, T^* g\, d\tau, \quad f, g \in M_+. \qquad (12)$$

Let Φ_1, Φ_2 be Young functions that are positive on $R_+ = (0,+\infty)$, $u, v, w \in M_+$, and let condition (7) holds. Let the operator $\Re_0$ be given by formula (9). Then the following inequalities are equivalent:

$$\exists c_1 \in R_+ : \quad \Phi_2^{-1}\left\{\int_{R_+} \Phi_2(wTf) u\, dt\right\} \le \Phi_1^{-1}\left\{\int_{R_+} \Phi_1(c_1 f) v\, dt\right\}, \quad f \in \Omega;$$

(13)

$$\exists c_3 \in R_+ : \Phi_2^{-1}\left\{\int_{R_+} \Phi_2\left(wT\Re_0^*(vf)\right) u\, dt\right\} \le \Phi_1^{-1}\left\{\int_{R_+} \Phi_1(c_3 f) v\, dt\right\},\ f \in M_+ \qquad (14)$$

Definition 3. A generalized Hardy Operator is an operator of the form

$$Kf(x)=\int_0^x k(x,t)f(t)dt, \qquad \mathrm{K}^* g(t)=\int_t^{+\infty} k(x,t)g(x)dx, \tag{15}$$

where

a) $k:\{(x,t)\in R^2 : 0<t<x<+\infty\}\to[0,+\infty)$;

b) $k(x,t)\geq 0$ is nondecreasing in x, nonincreasing in t;

c) $k(x,y)\leq D(k(x,t)+k(t,y))$, whenever $0\leq y\leq t<x<+\infty$ for some constant D. (16)

Proposition 3 ([1]). Let Φ_1,Φ_2 are N-function and $\Phi_1 \prec\prec \Phi_2$, and K be a generalized Hardy operator(15). Let a, b, υ and ω be nonnegative weight functions. Then there exists a constant $A>0$ such that

$$\Phi_2^{-1}\left(\int_0^{+\infty}\Phi_2(a\mathrm{K}f)\omega dx\right)\leq \Phi_1^{-1}\left(\int_0^{+\infty}\Phi_1(Afbdx)\upsilon\right)$$

for all nonnegative, measurable functions f if and only there exists a constant C such that

$$\Phi_2^{-1}\left(\int_r^{+\infty}\Phi_2\left(\frac{a(x)}{C}\left\|\frac{k(r\ ;.)\chi_{(0,r)}(.)}{\varepsilon\upsilon b}\right\|_{\psi_{1(\varepsilon\upsilon)}}\right)\omega(x)dx\right)\leq \Phi_1^{-1}\left(\frac{1}{\varepsilon}\right)$$

and

$$\Phi_2^{-1}\left(\int_r^{+\infty}\Phi_2\left(\frac{a(x)}{C}\left\|\frac{\chi_{(0,r)}(.)}{\varepsilon\upsilon b}\right\|_{\psi_{1(\varepsilon\upsilon)}}k(x\ ;r)\right)\omega(x)dx\right)\leq \Phi_1^{-1}\left(\frac{1}{\varepsilon}\right)$$

holds for $\varepsilon, r>0$.

Proposition 4 ([1]). Let Φ_1,Φ_2 are N-function and $\Phi_1 \prec\prec \Phi_2$, and K^* be a generalized Hardy operator (15). Let a, b, υ and ω be nonnegative weight functions. Then there exists a constant $A>0$ such that

$$\Phi_2^{-1}\left(\int_0^{+\infty}\Phi_2(a\mathrm{K}^*f)\omega dt\right)\leq \Phi_1^{-1}\left(\int_0^{+\infty}\Phi_1(Abf)\upsilon dt\right)$$

holds for all nonnegative, measurable functions f if and only there exists a constant C such that

$$\Phi_2^{-1}\left(\int_0^{r}\Phi_2\left(\frac{a(t)}{C}\left\|\frac{k(.\ ;r)\chi_{(\mathrm{r},+\infty)}(.)}{\varepsilon\upsilon b}\right\|_{\psi_{1(\varepsilon\upsilon)}}\right)\omega(t)dt\right)\leq \Phi_1^{-1}\left(\frac{1}{\varepsilon}\right)$$

and

$$\Phi_2^{-1}\left(\int_0^{r}\Phi_2\left(\frac{a(t)}{C}\left\|\frac{\chi_{(\mathrm{r},+\infty)}(.)}{\varepsilon\upsilon b}\right\|_{\psi_{1(\varepsilon\upsilon)}}k(r\ ;t)\right)\omega(t)dt\right)\leq \Phi_1^{-1}\left(\frac{1}{\varepsilon}\right)$$

holds for $\varepsilon, r>0$.

II. APPLICATIONS FOR HARDY-TYPE OPERATORS

Let us now state the main result of this section allowing us to reduce modular inequalities for operators on the cone Ω to modular inequalities for modified operators on the cone M_+.

$$\Im(f\,;x)=\int_x^{\infty} f(\tau)\,d\tau,\ \ \tau\in R_+. \tag{17}$$

Theorem 1. Let Φ_1,Φ_2 be N-function and $\Phi_1 \prec\prec \Phi_2$, w,u,v be positive weight functions, $\Im$ be Hardy-type operators (16), then there exists a constant $C>0$ such that

$$\Phi_2^{-1}\left\{\int_{R_+}\Phi_2\big(w(t)\Im f\big)u(t)\,dt\right\}\leq \Phi_1^{-1}\left\{\int_{R_+}\Phi_1\big(Cg\big)v\,dt\right\},\ \ f\in\Omega, \tag{18}$$

holds for all nonnegative, nonincreasing functions f if and only if there is a constant B such that all of the following inequalities hold for all ε, r > 0:

$$\Phi_2^{-1}\left\{\int_0^r\Phi_2\left(\frac{w(t)}{B}\left\|\frac{k(r,.)\chi_{(r,+\infty)}}{\varepsilon V}\right\|_{\psi_1(\varepsilon v)}\right)u(t)dt\right\}\leq \Phi_1^{-1}\left(\frac{1}{\varepsilon}\right), \tag{19}$$

holds for $\varepsilon, v>0$.

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Proof. 1. The purpose of the first step is to reduce estimate (14) to the estimate for the Hardy-type operator in the paper by Jim Quile Sun [1]. For the Hardy-type operator (19), using (11) and changing the order of integration, we obtain

$$\Im\left(\Re_0^*(vf;t)\right)=\int_t^\infty\left(\int_\tau^\infty \frac{f(x)v(x)}{V(x)}\,dx\right)d\tau,\ \tau\in R_+$$

$$\Im\left(\Re_0^*(vf;t)\right)=\int_t^\infty \frac{f(x)v(x)}{V(x)}\left(\int_t^x d\tau\right)dx,\ \tau\in R_+$$

$$\Im\left(\Re_0^*(vf;t)\right)=\int_t^\infty \frac{f(x)v(x)}{V(x)}(x-t)dx,\ \tau\in R_+$$

$$\Im\left(\Re_0^*(vf;t)\right)=\int_t^\infty k(x,t)g(x)\,dx,\ t\in R_+, \tag{20}$$

where

$$k(x,t)=x-t\,,\qquad g(x)=\frac{f(x)v(x)}{V(x)}\,,\qquad \sigma(t)=V(t)v^{-1}(t)\,. \tag{21}$$

we obtain the equivalence of (14) and (22), where (22) is of the form

$$\exists\, c_3\in R_+:\ \Phi_2^{-1}\left\{\int_{R_+}\Phi_2\left(w(t)\int_t^\infty k(x,t)g(x)\,dx\right)u(t)\,dt\right\}\le\ \Phi_1^{-1}\left\{\int_{R_+}\Phi_1\left(c_3\sigma\, g\right)v\,dt\right\},\ g\in M_+ \tag{22}$$

As a result, introducing the operator

$$\Im_0(g;t)=\int_t^\infty k(x,t)g(x)\,dx,\ t\in R_+ \tag{23}$$

we see that the kernel $k(x,t)$ is nonincreasing with respect to $x\in(t,\infty]$, and nondecreasing with respect to $t\in(0,x]$. The kernel also satisfies the triangle inequality

$$k(x,t)\le\ \mathrm{D}\left(\mathrm{k}(\mathrm{x},\mathrm{y})+k(y,t)\right),\qquad \mathrm{t}\le\mathrm{y}\le\mathrm{x}\,,$$

i.e., it is the kernel of the generalized Hardy operator $\Im_0$ in the Jim Quile Sun terminology [1]. Here the equivalence of the modular inequalities (14) and (22) holds.

2. We now pass to the proof of the equivalence of inequality (22) and the set of conditions (19). To this end, we use a known result due to Jim Quile Sun (see [1]) combined with the generalizations given in [4]. Denote

$$\Phi_2^{-1}\left\{\int_0^r\Phi_2\left(\frac{w(t)}{B}\left\|\frac{k(r,\,.)\chi_{(r,+\infty)}}{\varepsilon v\sigma}\right\|_{\psi_1(\varepsilon v)}\right)u(t)dt\right\}\le\ \Phi_1^{-1}\left(\frac{1}{\varepsilon}\right),$$

$$\Phi_2^{-1}\left\{\int_0^r\Phi_2\left(\frac{w(t)}{B}\left\|\frac{k(r,\,.)\chi_{(r,+\infty)}}{\varepsilon V}\right\|_{\psi_1(\varepsilon v)}\right)u(t)dt\right\}\le\ \Phi_1^{-1}\left(\frac{1}{\varepsilon}\right). \tag{24}$$

Thus, we have shown that (14) ⇐⇒ (22) ⇐⇒ (19).
Theorem 1 is proved.

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