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Dynamics of bio Economic and Optimal Harvesting in Multi Species Fish Population Dynamical System

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ABSTRACT

In this article, we propose a non-selective harvesting of a predator-prey scheme, utilising a rational harvesting quota rather than an indiscriminate one. In this model, the prey populations adhere to the principle of logistic growth. The successful reaction of predator's to prey density has been structured in such a way that any predator's realistic reaction to prey density is almost invariable as the prey population grows. The boundedness of the proposed model is verified. Eigenvalue procedure is used to analyze the existence of the consistent states as well as their local and global stabilities. The presence of bionomic equilibrium has been represented. Pontryagin's optimum theorem is then used to address the issue of deciding the best harvesting strategy. To investigate the impact of environmental variability on the harvested structure, we expect white noise stochastic perturbations. The numerical simulation confirmed our findings.

Keywords
Logistic growth, Bioeconomic equilibrium point, Optimal harvesting, Stochastic Stability, White noise.

Introduction

According to latest research, the interaction between predator and prey has become an important aspect to consider in the ecology framework model. This system has piqued the interest of biologists.

Population dynamics is the science which deals and describes the forces acting animal populations and how the populations react to these forces. Within fisheries ecology, our main aim is specific. What will happen to the fish population. Usually call it as stock when they are subjected to the specific external forces. What will happen to the stock means, size of the stock, growth, mortality, structure of the stock yield in other words net population? External forces mean, biotic environment such as prey-predators, competitors etc., and human activities.

In recent years, the purpose of fisheries management is to guarantee that catches from fish stock are ecologically feasible in the long term and benefits to fisheries and communities are maximized. Sea food is the main source of protein in the many parts of the world. So, stock sustainability is a universal requirement. In recent years the problems related to mathematical understanding of harvesting, bioeconomic yield, bioeconomic equilibrium states of multispecies fish population have been drawing an attention in researchers.

Maximum economic yield (MEY) and maximum sustainable yield (MSY) are the two critical concepts in optimum harvesting. Fisheries fall into the category of renewable natural resources. If the pace at which fish are

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harvested exceeds the rate at which they are reproduced, the fish stock will become extinct. MSY is a natural occurrence. MSY refers to the maximum amount of fish capture or yield that can be collected from a given system indefinitely without reducing the system's fish population (stock). In other terms, a capture ratio is said to be permanent if it is equivalent to the population growth rate and can be sustained indefinitely.

The maximum economic yield is the amount of yield that coincides with the degree of harvest or commitment that maximises the total returns from fishing in the long run. A MEY harvest is recommended because it is the capture amount that allows society to achieve its maximum potential.

Many scholars investigated either prey population harvesting [1, 2, 3, 4], predator population harvesting [5,6,7,8,9,10], or predator and prey population harvesting [11,12]. For example, in [1,12,13], the majority of predator-prey models with harvesting were correlated with economic issues such as the most benefit problem, taxation impact, and overall discounted net revenue problem.

In the last few decades, there has been an increase of study involvement in obtaining more detailed indicators of input impact in the issue of bioeconomic analysis of renewable resource exploitation, such as fisheries and forestry. Kar [14] and Chakraborty [15] investigated the prey-predator paradigm in fisheries, while Tapas et al.[16], Ganguli at el [17], Chaudhuri and Kar [18] investigated the bioeconomic component of the prey predator method and found that increasing harvesting activities result in population declines.

Bioeconomic modelling of habitant fisheries relationships was reported by Naomi S Foley [19] and colleagues. They attempted to explain the essence of the relations between the role of inhabitants and the economic activities they promote in this paper. Lowis W. Botsford investigated fisheries conservation shortcomings to achieve primary sustainability [20].

In the preset study, we considered the analytical model for one prey – two predator system with logistic growth rates and Holling type response. However, we harvested only predator population. Several types harvesting function have been studied particularly the following three types (1) Constant harvesting (2) Proportionate harvesting (3) Nonlinear harvesting. We have been considered the one prey and two predator system with non linear harvesting. We addressed the system's boundedness and the presence of a possible interior equilibrium condition, as well as local stability. The best harvesting method is often discussed using Pontryagin's optimum theorem. To investigate the impact of environmental variability on the harvested structure, we expect white noise stochastic perturbations. Numerical simulations are performed to confirm our analytical study.

Modeling of the problem

The dynamical structure is analysed in the following manner.

$$\frac{dx}{dt} = r_1 x\left(1-\frac{x}{K_1}\right) - \frac{mxy}{a+x} - \frac{nxz}{a+x}$$

$$\frac{dy}{dt} = r_2 y\left(1-\frac{y}{K_2}\right) + \frac{\beta mxy}{a+x} \tag{1}$$

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$$\frac{dz}{dt} = r_3 z\left(1 - \frac{z}{K_3}\right) + \frac{\gamma nxz}{a+x}$$

where x, y and z represents the sizes of the prey, predator-1 and predator-2 population at time t; K_1, environmental carrying capacity of the prey; K_2, environmental carrying capacity of the predator-1 and K_3 predator-2 carrying capacity; β, γ are conversion factors (we consider $(0 < \beta, \gamma < 1)$, since the whole biomass of the prey is not converted into predator biomass. m, n are maximal relative increases of predation, r_1, r_2, r_3 are the intrinsic growth rates of the prey, predator-1 and predator-2 respectively, a is Michaelis-Menten constant and $E = E(t)$, is effort function.

Fishery figures usually provide details under the heading fishing effort, which is calculated in units suitable for the fishery in question, according to Clark [21]. Let h denotes the rate at which a fish species is removed or harvested per unit of time. Almost always, the ratio of capture separated by commitment is used as a rough indicator of the total stock size of the fish community. Since the catch-per-unit-effort is relative to stock scale, we'll call this the grab per unit effort hypothesis, or that $h = cEx$ where E denotes effort and c is catchability coefficient is constant. This function has certain undesirable features, such as a random fish search, an equal probability of being captured with each fish, linear increase of h with E for a fixed x, and unbounded linear increase of h with x for a fixed E.

In the functional form, these limitations are completely eliminated, $h(t) = \frac{[c_1 E(t) x(t)]}{[l_1 E(t) + l_2 x(t)]}$ Proposed first by Clark. It is noticed that $h \to (c_1 / l_1)x$ as $E \to \infty$ for a fixed value of x and $h \to (c_1 / l_2)E$ as $x \to \infty$ for a fixed value of E.

The parameter l_1 is proportional to the ratio of stock level to grab speed at higher levels of effort, and the parameter l_2 is proportional to the ratio of effort level to seize rate at higher stock levels. All of the parameters are assumed to be positive.

The term "predation" has traditionally been described as mxy. In a wider sense, mxy may be thought of as a predator's tropic system or practical response to prey density. We don't use the term "predation" in this context since it means that as the prey population expands endlessly with a small and fixed predator population, predation tends to infinity. As a consequence, predation is described as

$$\frac{mxy}{a+x} \text{ as } \underset{x\to\infty}{Lt} \frac{mxy}{a+x} = my \text{ and } \frac{\gamma xz}{a+x} \text{ as } \underset{x\to\infty}{Lt} \frac{\gamma xz}{a+x} = \gamma z$$

Assuming that both predator-1 and predator-2 are subjected to a combined harvesting effort E, we may written as

$$\frac{dx}{dt} = r_1 x\left(1 - \frac{x}{K_1}\right) - \frac{mxy}{a+x} - \frac{nxz}{a+x}$$

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$$\frac{dy}{dt} = r_2 y\left(1-\frac{y}{K_2}\right)+\frac{\beta mxy}{a+x}-\frac{c_1Ey}{l_1E+l_2y}$$

(2)

$$\frac{dz}{dt} = r_3 z\left(1-\frac{z}{K_3}\right)+\frac{\gamma nxz}{a+x}-\frac{c_2Ez}{l_3E+l_4y}$$

where c_1 and c_2 coefficients of catchability for the two predators. The catch rate functions are taken as $\frac{c_1Ey}{l_1E+l_2y}$ and $\frac{c_2Ez}{l_3E+l_4y}$. Here, l_1, l_2, l_3, l_4 are positive constants. The ratio of $\frac{r_{i+1}}{c_i}$, (where $i = 1, 2$) of the biotic potential (r_i) to the coefficient of catchability (c_i) is known as the species biotechnical productivity. The structure of the manipulated mechanism (equation (2)) will now be investigated.

Boundedness of the scheme

Theorem (1): The solutions of the system (2) which start in R_2^+ are uniformly bounded.

Proof: We define the function $\Psi(x,y,z) = x+\frac{1}{\beta}y+\frac{1}{\gamma}z$ (3)

The derivative of (3) w.r.t t is

$$\Psi'(x,y,z) = r_1x\left(1-\frac{x}{K_1}\right)+\frac{r_2}{\beta}y\left(1-\frac{y}{K_2}\right)-\frac{c_1Ey}{\beta(l_1E+l_2y)}+\frac{r_3}{\gamma}z\left(1-\frac{z}{K_3}\right)-\frac{c_2Ez}{\gamma(l_3E+l_4y)}$$

For any $u > 0$, we get

$$\Psi'+u\Psi = r_1x\left(1-\frac{x}{K_1}\right)+\frac{r_2}{\beta}y\left(1-\frac{y}{K_2}\right)-\frac{c_1Ey}{\beta(l_1E+l_2y)}+\frac{r_3}{\gamma}z\left(1-\frac{z}{K_3}\right)$$
$$-\frac{c_2Ez}{\gamma(l_3E+l_4y)}+ux+\frac{u}{\beta}y+\frac{u}{\gamma}z$$

$$\Psi'+u\Psi < (r_1+u)x+\left(\frac{r_2+u}{\beta}\right)y+\left(\frac{r_3+u}{\gamma}\right)z$$

$\Psi'+u\Psi < K_1D_1+K_2D_2+K_3D_3 < L$, as $0 \le x \le K_1, 0 \le y \le K_2, 0 \le z \le K_3$

Where $L = Max_{\{x,y,z\}}\{K_1D_1, K_2D_2, K_3D_3\}$ and $D_1 = r_1+u, D_2 = \frac{r_2+u}{\beta}, D_3 = \frac{r_3+u}{\gamma}$

By the theory of differential inequality, we obtained

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$$0 < \Psi\left(x, y, z\right) < \frac{L}{u}\left(1 - e^{-ut}\right) + \Psi\left(x(0), y(0), z(0)\right)e^{-ut}$$

As $t \to \infty$, the above inequality becomes $0 < \Psi < \frac{L}{u}$

Hence, all the solutions of the system (2) are confined in the region $\Omega = \left\{\left(x, y, z\right) \in R_3^+ : \Psi = \frac{L}{u} + \varepsilon, \text{any } \varepsilon > 0\right\}$

Equilibria of the scheme

Here we considering only positive equilibrium point and it is very difficult to solve the equations $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0 \,\&\, \frac{dz}{dt} = 0$ in equation (2). Thus, we can assume all the three values are non-negative. Therefore, interior equilibrium point is existing.

Local Stability Analysis of the scheme

The Jacobian matrix of the system (2) is given by

$$J = \begin{bmatrix} -\frac{r_1 x}{K_1} + \frac{mxy}{\left(a+x\right)^2} + \frac{nxz}{\left(a+x\right)^2} & -\frac{mx}{a+x} & -\frac{nx}{a+x} \\ \frac{\beta may}{\left(a+x\right)^2} & -\frac{r_2 y}{K_2} + \frac{c_1 l_2 Ey}{\left(l_1 E + l_2 y\right)^2} & 0 \\ \frac{\gamma naz}{\left(a+x\right)^2} & 0 & -\frac{r_3 z}{K_3} + \frac{c_2 l_4 Ez}{\left(l_3 E + l_4 z\right)^2} \end{bmatrix}$$

The characteristic equation is given by $\lambda^3 + P_1\lambda^2 + P_2\lambda + P_3 = 0$

where $P_1 = \frac{-r_1 x}{K_1} + \frac{mxy}{\left(a+x\right)^2} + \frac{nxz}{\left(a+x\right)^2} - \frac{r_2 y}{K_2} + \frac{c_1 l_2 Ey}{\left(l_1 E + l_2 y\right)^2} - \frac{r_3 z}{K_3} + \frac{c_2 l_4 Ez}{\left(l_3 E + l_4 z\right)^2}$

$$\frac{r_1 r_2 xy}{K_1 K_2} - \frac{r_1 c_1 l_2 Exy}{K_1\left(l_1 E + l_2 y\right)^2} - \frac{r_2 ymxy}{K_2\left(a+x\right)^2} - \frac{r_2 ynxz}{K_2\left(a+x\right)^2} + \frac{mc_1 l_2 Exy^2}{\left(a+x\right)^2\left(l_1 E + l_2 y\right)^2} + \frac{nc_1 l_2 Eyz}{\left(a+x\right)^2\left(l_1 E + l_2 y\right)^2} \tag{4}$$

$$\frac{r_2 r_3 yz}{K_2 K_3} - \frac{r_2 c_2 l_4 Eyz}{K_2\left(l_3 E + l_4 z\right)^2} - \frac{r_3 c_1 l_2 Eyz}{K_3\left(l_1 E + l_2 y\right)^2} + \frac{c_1 c_2 l_2 l_4 E^2 yz}{\left(l_1 E + l_2 y\right)^2\left(l_3 E + l_4 z\right)^2} \tag{5}$$

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$$\frac{r_1 r_3 xz}{K_1 K_3} - \frac{m r_3 xyz}{K_3 (a+x)^2} - \frac{n r_3 xz^2}{K_3 (a+x)^2} - \frac{r_1 c_2 l_4 Exz}{K_1 (l_3 E + l_4 z)^2} + \frac{m c_2 l_4 Exyz}{(a+x)^2 (l_3 E + l_4 z)^2} + \frac{n c_2 l_4 Exz^2}{(a+x)^2 (l_3 E + l_4 z)^2}$$

(6)

$$\frac{\beta m^2 axy}{(a+x)^3} + \frac{\gamma n^2 axz}{(a+x)^3}$$

(7)

$$P_2 = \text{equatins}(4)+(5)+(6)+(7)$$

$$P_3 = |J| = \left(-\frac{r_3 z}{K_3} + \frac{c_2 l_4 Ez}{(l_3 E + l_4 z)^2}\right)\left[\left(-\frac{r_1 x}{K_1} + \frac{mxy}{(a+x)^2} + \frac{nxz}{(a+x)^2}\right)\left(-\frac{r_2 y}{K_2} + \frac{c_1 l_2 Ey}{(l_1 E + l_2 y)^2}\right) + \frac{\beta m^2 axy}{(a+x)^3}\right] + \frac{n^2 \gamma axz}{(a+x)^3}\left[-\frac{r_2 y}{K_2} + \frac{c_1 l_2 Ey}{(l_1 E + l_2 y)^2}\right]$$

Therefore, given system is locally asymptotically stable if $P_1 > 0, P_2 > 0, P_3 > 0$ and $P_1 P_2 - P_3 > 0.$

Bionomic equilibrium

The term "bionomic equilibrium" refers to the combination of biological and economic equilibrium which is given by $\dot{x} = \dot{y} = \dot{z} = 0.$ when the net revenue gained from the sale of harvested biomass (TR) approaches the total cost (TC) of harvesting effort, the economic equilibrium is said to have been reached.

Allow for a constant $C,$ fishing cost per unit effort, as well as a constant $p_1,$ prey biomass of prey and p_2 predator animals and it is constant.

For every given moment, the economic rent (net revenue) is calculated.

$$\pi(y,z,E) = \left(\frac{p_1 c_1 y}{l_1 E + l_2 y} + \frac{p_2 c_2 z}{l_3 E + l_4 z} - C\right)E$$

(8)

While the harvesting cost per unit effort is not a constant, we regard it as such for the sake of convenience. Now,

$$\frac{dy}{dt} = 0 \Rightarrow r_2\left(1 - \frac{y}{K_2}\right) + \frac{\beta mx}{a+x} = \frac{c_1 E}{l_1 E + l_2 y}$$

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$$\Rightarrow c_1E = l_1E\left[r_2\left(1-\frac{y}{K_2}\right)+\frac{\beta mx}{a+x}\right]+l_2y\left[r_2\left(1-\frac{y}{K_2}\right)+\frac{\beta mx}{a+x}\right]$$

$$E=\frac{l_2y\left[r_2\left(1-\frac{y}{K_2}\right)+\frac{\beta mx}{a+x}\right]}{C_1-l_1\left[r_2\left(1-\frac{y}{K_2}\right)+\frac{\beta mx}{a+x}\right]}$$

Here E is positive when

$$\frac{C_1}{l_1}>r_2\left(1-\frac{y}{K_2}\right)+\frac{\beta mx}{a+x}. \tag{9}$$

Similarly

$$\frac{dz}{dt}=0\Rightarrow r_3\left(1-\frac{z}{K_3}\right)+\frac{\gamma nx}{a+x}=\frac{c_2E}{l_3E+l_4z}$$

$$c_2E=l_3E\left[r_3\left(1-\frac{z}{K_3}\right)+\frac{\gamma nx}{a+x}\right]+l_4z\left[r_3\left(1-\frac{z}{K_3}\right)+\frac{\gamma nx}{a+x}\right]$$

$$E=\frac{l_4z\left[r_3\left(1-\frac{z}{K_3}\right)+\frac{\gamma nx}{a+x}\right]}{C_2-l_3\left[r_3\left(1-\frac{z}{K_3}\right)+\frac{\gamma nx}{a+x}\right]}$$

Here E is positive when

$$\frac{C_2}{l_3}>r_3\left(1-\frac{z}{K_3}\right)+\frac{\gamma nx}{a+x} \tag{10}$$

As a result, the positive equilibrium solution is found at a certain point on the curve. $\frac{l_2y\left[r_2\left(1-\frac{y}{K_2}\right)+\frac{\beta mx}{a+x}\right]}{C_1-l_1\left[r_2\left(1-\frac{y}{K_2}\right)+\frac{\beta mx}{a+x}\right]}=\frac{l_4z\left[r_3\left(1-\frac{z}{K_3}\right)+\frac{\gamma nx}{a+x}\right]}{C_2-l_3\left[r_3\left(1-\frac{z}{K_3}\right)+\frac{\gamma nx}{a+x}\right]}$ (11)

Where $0\le y\le K_2$, $0\le z\le K_3$

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The bionomic equilibrium $\left(x_\infty, y_\infty, z_\infty\right)$ of the open access fishery is determined by equation (11) together with the condition

$$\pi = TR - TC$$
$$= \left(\frac{p_1c_1y}{l_1E+l_2y}+\frac{p_2c_2z}{l_3E+l_4z}-c\right)E = 0 \qquad (12)$$
$$\frac{p_1c_1y}{l_1E+l_2y}+\frac{p_2c_2z}{l_3E+l_4z}-c=0$$

Optimal harvesting procedure

The central issue in commercial renewable resource extraction is determining the best trade-off between existing and potential harvests. This dilemma is much too difficult to address whether the global, societal, and metaphysical aspects of it are included. However, if we just see the dilemma from an economic perspective, we could use the traditional time discounting methodology to answer concerns about inter-temporal economic benefits. The concept of discounting economic gains (or costs) over time is common in business management. Clark [21] demonstrates that the idea of optimising long-term economic rent is impractical since it entails setting the discount rate to zero.

$$J = \int_0^\infty \pi\left(x,y,z,E,t\right)e^{-\delta t}dt \qquad (13)$$

$$H = \left(\frac{p_1c_1y}{l_1E+l_2y}+\frac{p_2c_2z}{l_3E+l_4z}-c\right)Ee^{-\delta t}+\lambda_1\left[r_1x\left(1-\frac{x}{K_1}\right)-\frac{mxy}{a+x}-\frac{\delta nxz}{a+x}\right]$$
$$+\lambda_2\left[r_2y\left(1-\frac{y}{K_2}\right)+\frac{\beta mxy}{a+x}-\frac{c_1Ey}{l_1E+l_2y}\right]+\lambda_3\left[r_3z\left(1-\frac{z}{K_3}\right)+\frac{\gamma nxz}{a+x}-\frac{c_2Ez}{l_3E+l_4y}\right] \qquad (14)$$

The adjoint equations are given by

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x} = -\lambda_1\left[r_1\left(1-\frac{2x}{K_1}\right)-\frac{may}{\left(a+x\right)^2}-\frac{naz}{\left(a+x\right)^2}\right]-\lambda_2\frac{\beta amr}{\left(a+x\right)^2}-\lambda_3\frac{\gamma anz}{\left(a+x\right)^2} \qquad (15)$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y} = \lambda_1\frac{mx}{\left(a+x\right)^2}-\lambda_2\left[r_2\left(1-\frac{2y}{K_2}\right)+\frac{\beta mx}{\left(a+x\right)^2}-\frac{c_1l_1E^2}{\left(l_1E+l_2y\right)^2}\right]-\frac{P_1c_1l_1E^2}{\left(l_1E+l_2y\right)^2}e^{-\delta t} \qquad (16)$$

$$\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial z} = \lambda_1\frac{nx}{a+x}-\lambda_3\left[r_3\left(1-\frac{2z}{K_3}\right)+\frac{\gamma nx}{a+x}-\frac{c_2l_3E^2}{\left(l_3E+l_4z\right)^2}\right]-\frac{P_2c_2l_3E^2}{\left(l_3E+l_4z\right)^2}e^{-\delta t} \qquad (17)$$

We concentrate on locating the problem's optimum equilibrium solution so that we may accept it.

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$$E=\frac{l_2 y\left[r_2\left(1-\frac{y}{K_2}\right)+\frac{\beta mx}{a+x}\right]}{C_1-l_1\left[r_2\left(1-\frac{y}{K_2}\right)+\frac{\beta mx}{a+x}\right]}=\frac{l_4 z\left[r_3\left(1-\frac{z}{K_3}\right)+\frac{\gamma nx}{a+x}\right]}{C_2-l_3\left[r_3\left(1-\frac{z}{K_3}\right)+\frac{\gamma nx}{a+x}\right]} \tag{18}$$

By using equations (18), (15), (16) and (17), we can obtain the expression

$$\left[D^3+\left(A_1+B_2+C_2\right)D^2+\left(A_1B_2+B_1A_2+A_1C_2+C_2B_2-C_1A_3\right)D+A_1B_2C_2+A_2B_1C_2-A_3B_2C_1\right]\lambda_2$$
$$=\left[B_3\delta^2-A_1B_3\delta-B_3C_2\delta+A_1B_3C_2-A_3B_3C_1-A_3B_1C_3\right]e^{-\delta t} \tag{19}$$

where $D=\frac{d}{dt}$, $A_1=r_1\left(1-\frac{2x}{K_1}\right)-\frac{may}{(a+x)^2}-\frac{naz}{(a+x)^2}$, $A_2=\frac{\beta amr}{(a+x)^2}$, $A_3=\frac{\gamma anz}{(a+x)^2}$

$B_1=\frac{mx}{(a+x)^2}$, $B_2=r_2\left(1-\frac{2y}{K_2}\right)+\frac{\beta mx}{(a+x)^2}-\frac{c_1l_1E^2}{(l_1E+l_2y)^2}$ $\quad B_3=-\frac{P_1c_1l_1E^2}{(l_1E+l_2y)^2}$

$C_1=\frac{nx}{a+x}$, $C_2=r_3\left(1-\frac{2z}{K_3}\right)+\frac{\gamma nx}{a+x}-\frac{c_2l_3E^2}{(l_3E+l_4z)^2}$, $C_3=-\frac{P_2c_2l_3E^2}{(l_3E+l_4z)^2}$

The auxiliary equation of (18) is given by

$$\mu^3+b_1\mu^2+b_2\mu+b_3=P_1e^{-\delta t} \tag{20}$$

where

$$P_1=B_3\delta^2-A_1B_3\delta-B_3C_2\delta+A_1C_2B_3-A_3B_3C_1-A_3B_1C_3 \tag{21}$$

Let $\mu_1,\ \mu_2,\ \mu_3$ be the roots of the equation (20), then

$$\lambda_2=Ae^{\mu_1 t}+Be^{\mu_2 t}+Ce^{\mu_3 t}+\frac{P_1}{Q}e^{-\delta t} \text{ as } t\to\infty,\ \lambda_2(t)e^{\delta t}=\frac{P_1}{Q}\text{ constant} \tag{22}$$

Similarly, $\lambda_3(t)e^{\delta t}=\frac{P_2}{Q}$ is also constant. (23)

where, $Q=-\delta^3+b_1\delta^2-b_2\delta+b_3$ (24)

Hence the shadow price $\lambda_2(t)e^{\delta t}$ remains bounded as $t\to\infty$ iff $A=B=C=0$ and then

$\lambda_2(t)e^{\delta t}=\frac{P_1}{Q}=\text{constant}$.

Similarly, we get $\lambda_3(t)e^{\delta t}=\frac{P_2}{Q}=\text{constant}$.

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$$\frac{\partial H}{\partial E}=\left(\frac{p_1c_1y}{l_1E+l_2y}+\frac{p_2c_2z}{l_3E+l_4z}-C\right)e^{-\delta t}-\left[\frac{c_1l_2y^2}{\left(l_1E+l_2y\right)^2}\right]\lambda_2-\left[\frac{c_2l_4z^2}{\left(l_3E+l_4z\right)^2}\right]\lambda_3=0$$

$$\left[\frac{c_1l_2y^2}{\left(l_1E+l_2y\right)^2}\right]\lambda_2+\left[\frac{c_2l_4z^2}{\left(l_3E+l_4z\right)^2}\right]\lambda_3=e^{-\delta t}\frac{\partial \pi}{\partial E} \tag{25}$$

The discounted valuation of potential earnings per unit effort at the steady state effort stage as seen on the right hand side.

By substituting the values of $\lambda_2(t),\ \lambda_3(t)$ in the equation (25), we get

$$\left\{p_1-\frac{P_1}{Q}\right\}\left[\frac{c_1l_2y^2}{\left(l_1E+l_2y\right)^2}\right]+\left\{p_2-\frac{P_2}{Q}\right\}\left[\frac{c_2l_4z^2}{\left(l_3E+l_4z\right)^2}\right]=C. \tag{26}$$

The equation (25) with the equation (8) gives the optimal equilibrium populations

$x=x_\delta,\ y=y_\delta,\ z=z_\delta$

where $\delta\to\infty$ the equation (26) leads to the result

$\frac{c_1l_2y^2}{\left(l_1E+l_2y\right)^2}+\frac{c_2l_4z^2}{\left(l_3E+l_4z\right)^2}=C,$ which implies $\frac{\partial \pi}{\partial E}\left(x_\infty, y_\infty, z_\infty, E\right)=0$

As a result, at an infinite discount rate, the value of potential earnings per unit effort vanishes. Using equation (26) as a guide, we arrive at

$$\frac{\partial \pi}{\partial E}=\left(\frac{p_1c_1l_2y^2}{\left(l_1E+l_2y\right)^2}+\frac{p_2c_2l_4z^2}{\left(l_3E+l_4z\right)^2}-C\right)=\frac{P_1}{Q}\frac{c_1l_2y^2}{\left(l_1E+l_2y\right)^2}+\frac{P_2}{Q}\frac{c_2l_4z^2}{\left(l_3E+l_4z\right)^2}$$

As each of P_1, P_2 is $o(\delta)$ where Q is $o\left(\delta^2\right)$, we can find that $\frac{\partial \pi}{\partial E}$ is $o\left(\delta^{-1}\right)$. Hence $\frac{\partial \pi}{\partial E}$ is a decreasing function of $\delta\geq 0$.

As a result, we may infer that $\delta=0$ contributes to the maximization of $\frac{\partial \pi}{\partial E}$

Stochastic Analysis

Environmental noise affects the dynamics of interacting populations, and several scholars have examined the dynamical activities of stochastic ecological interacting structures [22, 24]. R.M May [23] discovered that the parameters of stochastic systems oscillate about their mean values all the time, and that the solutions do as well. As a result of enabling stochastic variations in the variables x, y about their positive equilibrium value E^*, the obligatory mutualism scheme transforms into a stochastic differential equation (SDE).

$$dx=\left(r_1x\left(1-\frac{x}{K_1}\right)-\frac{mxy}{a+x}-\frac{nxz}{a+x}\right)dt+\sigma_1\left(x-x^*\right)d\xi_t^1$$

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$$dy = \left(r_2 y\left(1-\frac{y}{K_2}\right) + \frac{\beta mxy}{a+x} - \frac{c_1 Ey}{l_1E + l_2 y}\right)dt + \sigma_2\left(y - y^*\right)d\xi_t^2$$

$$dz = \left(r_3 z\left(1-\frac{z}{K_3}\right) + \frac{\gamma nxz}{a+x} - \frac{c_2 Ez}{l_3E + l_4 y}\right)dt + \sigma_3\left(z - z^*\right)d\xi_t^3 \quad (27)$$

Where $\sigma_i, i = 1,2,3$ are real constants, $\xi_t^i = \xi_i\left(t\right), i = 1,2,3$ are independent standard Wiener processes. To investigate the stochastic stability of E^*, consider the linear system (27) of around E^* as follows:

$$du(t) = f(u(t))dt + g(u(t))d\xi(t) \quad (28)$$

where $u(t) = col(u_1(t), u_2(t), u_3(t))$; $f(u(t)) = Ju(t)$; $g(u) = \begin{bmatrix} \sigma_1 u_1 & 0 & 0 \\ 0 & \sigma_2 u_2 & 0 \\ 0 & 0 & \sigma_3 u_3 \end{bmatrix}$;

$d\xi\left(t\right) = col(\xi_1(t), \xi_2(t), \xi_3(t))$; $u_1 = x - x^*$; $u_2 = y - y^*$; $u_3 = z - z^*$.

Let $U = \left\{\left(t \ge t_0\right) \times R^n, t_0 \in R^+\right\}$. Hence $V \in C_3^0(U)$ is a continuous function w.r.t t and a twice continuously differentiable function w. r. t to u , we have

$$LV(t,u) = \frac{\partial V(t,u)}{\partial t} + f^T(u)\frac{\partial V(t,u)}{\partial u} + \frac{1}{2}Tr\left(g^T(u)\frac{\partial^2 V(t,u)}{\partial u^2}g(u)\right) \quad (29)$$

where $\frac{\partial V}{\partial u} = Col\left(\frac{\partial V}{\partial u_1}, \frac{\partial V}{\partial u_1}\right)$; $\frac{\partial^2 V(t,u)}{\partial u^2} = \frac{\partial^2 V}{\partial u_j \partial u_i}; i, j = 1,2,3$ and T represents transposition

Theorem (2): If there exists a function $V(u,t) \in C_2^0(U)$ satisfying the following

$$K_1 |u|^p \le V(t,u) \le K_2 |u|^p ; LV(t,u) \le -K_3 |u|^p \;, K_i > 0 \;,\; p > 0 \quad (30)$$

Then the trivial solution of (8.4) is exponentially p-stable for $t \ge 0$.

Note that, if in (30), p = 2, then the trivial solution of (28) is globally asymptotically stable [25].

Theorem (3): Suppose that $\left(\left(\frac{r_1 x}{K_1} - \frac{mxy}{\left(a+x\right)^2} - \frac{nxz}{\left(a+x\right)^2}\right) - \frac{1}{2}\sigma_1^2\right) > 0$, $\frac{r_2 y}{K_2} - \frac{c_1 l_2 Ey}{\left(l_1E + l_2 y\right)^2} - \frac{1}{2}\sigma_2^2 > 0$ and $\frac{r_3 z}{K_3} - \frac{c_2 l_4 Ez}{\left(l_3E + l_4 z\right)^2} - \frac{1}{2}\sigma_3^2 > 0$ then the zero solution of (28) is asymptotically mean square stable.

Proof: Let us consider the Lyapunov function $V(u) = \frac{1}{2}\left(w_1 u_1^2 + w_2 u_2^2\right), w_i > 0 \in \Re$ (31)

The inequalities in (30)are true when p = 2 and we have

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$$
\begin{aligned}
LV(u) = {} & w_1\left(\left(-\frac{r_1 x}{K_1}+\frac{mxy}{(a+x)^2}+\frac{nxz}{(a+x)^2}\right)u_1+\left(-\frac{mx}{a+x}\right)u_2+\left(-\frac{nx}{a+x}\right)u_3\right)u_1 \\
& + w_2\left(\frac{\beta may}{(a+x)^2}u_1+\left(-\frac{r_2 y}{K_2}+\frac{c_1 l_2 Ey}{(l_1E+l_2y)^2}\right)u_2\right)u_2 \\
& + w_3\left(\frac{\gamma naz}{(a+x)^2}u_1+\left(-\frac{r_3 z}{K_3}+\frac{c_2 l_4 Ez}{(l_3E+l_4z)^2}\right)u_3\right)u_3 \\
& + \frac{1}{2}Tr\left[g^T(u)\frac{\partial^2 V}{\partial u^2}g(u)\right]
\end{aligned} \tag{32}
$$

We can clearly notice that $\frac{\partial^2 V}{\partial u^2}=\begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{pmatrix}$ and hence

$$
g^T(u)\frac{\partial^2 V}{\partial u^2}g(u)=\begin{bmatrix} w_1\sigma_1^2 u_1 & 0 & 0 \\ 0 & w_2\sigma_2^2 u_2 & 0 \\ 0 & 0 & w_3\sigma_3^2 u_3 \end{bmatrix}
$$

with $\frac{1}{2}Tr\left[g^T(u)\frac{\partial^2 V}{\partial u^2}g(u)\right]=\frac{1}{2}\left[w_1\sigma_1^2u_1^2+w_2\sigma_2^2u_2^2++w_3\sigma_3^2u_3^2\right]$

(33)

If in (32) we choose $\frac{\beta may}{(a+x)^2}w_2=\left(-\frac{mx}{a+x}\right)w_1$, then from (33), we have

$$
\begin{aligned}
LV(u) = {} & -w_1\left(\left(\frac{r_1 x}{K_1}-\frac{mxy}{(a+x)^2}-\frac{nxz}{(a+x)^2}\right)-\frac{1}{2}\sigma_1^2\right)u_1^2 - w_2\left(\frac{r_2 y}{K_2}-\frac{c_1 l_2 Ey}{(l_1E+l_2y)^2}-\frac{1}{2}\sigma_2^2\right)u_2^2 \\
& -w_3\left(\frac{r_3 z}{K_3}-\frac{c_2 l_4 Ez}{(l_3E+l_4z)^2}-\frac{1}{2}\sigma_3^2\right)u_3^2<0
\end{aligned}
$$

As indicated by ***Theorem (3)*** the proof is completed.

Numerical Simulations

Example1: Taking the parameter values in system (2) are

$r_1=1.5; r_2=0.5; r_3=0.4; K_1=65; K_2=66; K_3=5; m=0.905; \beta=0.815; \gamma=0.95;$
$a=10; n=0.62; c_1=0.214; c_2=0.037; E=241; l_1=0.451; l_2=0.125; l_3=0.52; l_4=0.05;$

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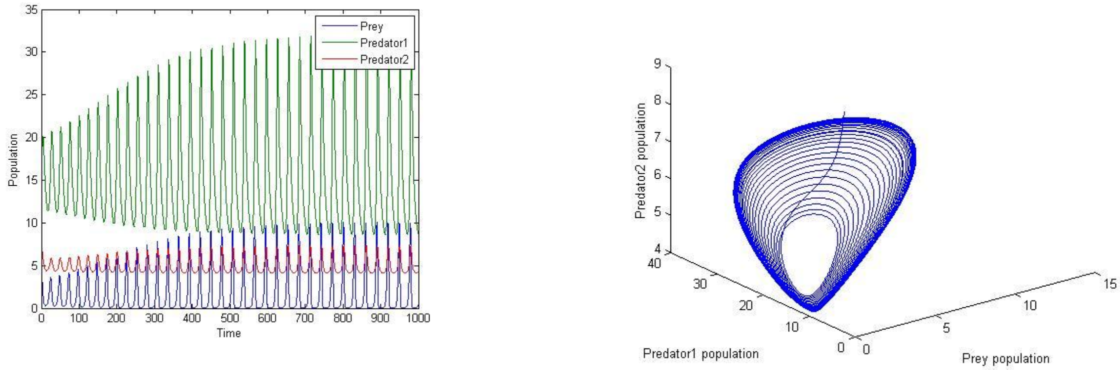

Fig1. The trajectories and phase graphs of system (2) with respect to the above parameter values and positive equilibrium is asymptotically stable.

Example2 By taking the above parameter values as same and for different values of white noise intensities in system (27), we can get the following graphs (Fig2, Fig3 and Fig4) and for these parameter values the system (27) satisfies the Theorem (3) conditions. From this we can observe that if we increase intensities of white noise then we get rule less oscillations when the trajectories are converging to the equilibrium point.

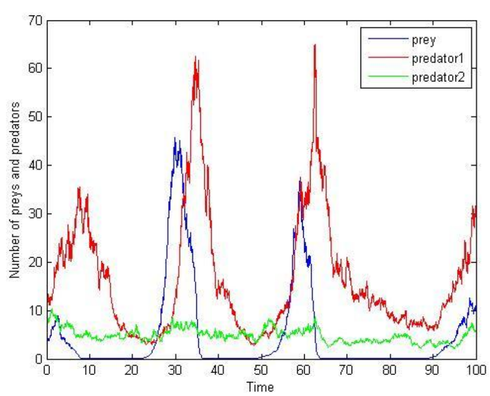

Fig2: For $\sigma_1 = \sigma_2 = \sigma_3 = 0.2$

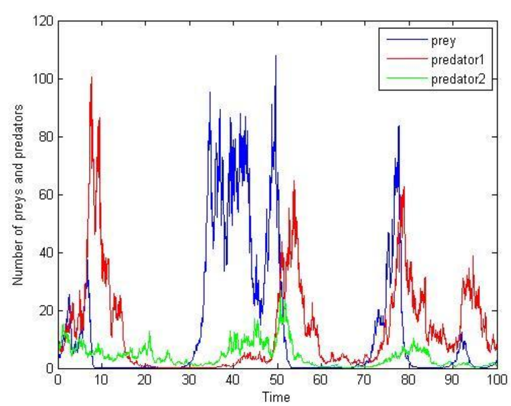

Fig3: For $\sigma_1 = \sigma_2 = \sigma_3 = 0.5$

Conclusions

The current research looks at a problem involving one prey and two predators, as well as nonlinear harvesting in the predator community. First, we established that the solutions to the model system under consideration are bounded. The local asymptotic stability of the interior equilibrium point was then considered. After that, the proposed system's bionomic (biological as well as economic) equilibrium is investigated. The convergence of the zero benefit line and the biological equilibrium line is at these points. The optimum harvesting strategy is then investigated using Pontryagin's maximal theorem. The authors hope to achieve some potentially useful results by using the proposed mathematical model and dynamical analysis to address efficient harvesting methods and sustainability mechanisms of harvested prey-predator fishery systems of nonlinear harvesting. Furthermore, the theoretical findings may be useful for administrative entities formulating regulatory strategies to maintain economic optimality while maintaining global sustainability.

Using a suitable Lyapunov function, we have obtained the requirement for asymptotic stability of positive equilibrium point in mean square sense for the stochastic variant of the model system. These conditions depend upon $\sigma_1, \sigma_2 \& \sigma_3$ and the parameters of the model system.

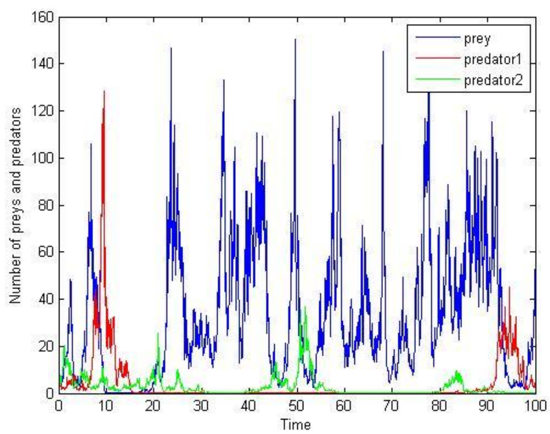

Fig4: For $\sigma_1 = \sigma_2 = \sigma_3 = 0.9$

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